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Euler–Lagrange and Hamilton equations for non-holonomic systems in field theory

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Abstract

A generalization of the concept of a system of non-holonomic constraints to fibred manifolds with n-dimensional bases is considered. Motion equations in both Lagrangian and Hamiltonian settings for systems subjected to such constraints are investigated. Regularity conditions for the existence of a non-holonomic Legendre transformation, and the corresponding formulae for Hamiltonian and momenta are found. In particular, Lagrangian constraints and semi-holonomic constraints, and simplifications arising in this case are discussed.

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1. Introduction

Recently, the geometry of non-holonomic systems in mechanics, inspired by the work of Chetaev [4], has been intensively studied. Among others, geometric structures connected with non-holonomic constraints in jet bundles have been described, and constrained systems have been considered as defined directly on constraint submanifolds (i.e., with 'eliminated Lagrange multipliers'). Within this setting, constrained Euler–Lagrange equations and constrained Hamilton equations have been found, a constraint Legendre transformation has been proposed, and symmetries of constrained systems have been studied (see, e.g., [2, 3, 6, 8, 13–15, 18–20, 22–25, 27–29] and references therein). Contrary to this situation, only a few pioneer works deal with constraints and constrained equations in field theory, i.e., for partial differential equations (see [21] for vakonomic-type constraints, and [1, 17] for constraints of non-holonomic type).

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This paper aims to be a contribution to developments of a mathematical formulation of a 'non-holonomic field theory'. We leave aside a discussion on applications of the theory which one could possibly search within some problems of field theories or continuum mechanics. In fact, at the moment, very little is known on this point; even in the case of (classical and higher order) mechanics there is still a shortage of concrete examples concerned with 'non-classical' constraints (e.g., non-linear in velocities or depending on higher derivatives). Furthermore, the role of vakonomic and non-holonomic constraints in applications in field theory is still far from being known and well understood (cf [21] for a discussion on this point).

We consider a fibred manifold $\pi : Y \to X$ with $\dim X = n$ and $\dim Y = m + n$, i.e., m denotes the fibre dimension, and its 1-jet prolongation J^1Y, with local fibred coordinates, is denoted by $\left(x^i, y^\sigma, y^\sigma_j\right)$, where $1 \leqslant i, j \leqslant n$, and $1 \leqslant \sigma \leqslant m$. A system of K *non-holonomic constraints* is defined to be a *submanifold* Q of codimension K in J^1Y fibred over Y, and locally is given by a system of K (linearly independent) *first-order PDE*

$$f^\alpha\left(x^i, y^\nu, y^\nu_j\right) = 0, \quad 1 \leqslant \alpha \leqslant K, \tag{1.1}$$

satisfying the rank condition

$$\operatorname{rank}\left(\frac{\partial f^\alpha}{\partial y^\nu_j}\right) = K. \tag{1.2}$$

It turns out that the family of possible non-holonomic constraints for partial differential equations is richer than that for ordinary differential equations (cf [17]). In particular, there is an interesting class of the so-called *π-adapted constraints*, which can be viewed as a non-trivial 'multi-variable' generalization of non-holonomic constraints of classical mechanics. They are locally defined by a system of $K = kn$ first-order partial differential equations in the normal form,

$$f^a_j \equiv y^{m-k+a}_j - g^a_j\left(x^i, y^\sigma, y^1_l, \ldots, y^{m-k}_l\right) = 0, \qquad 1 \leqslant a \leqslant k < m, \qquad 1 \leqslant j \leqslant n, \tag{1.3}$$

satisfying the (additional) rank condition

$$\operatorname{rank}\left(\frac{\partial f^a_j}{\partial y^\sigma_i}\right) = \text{const} < m, \qquad \text{where} \quad (a, j, i) \text{ label rows and } \sigma \text{ label columns}. \tag{1.4}$$

Remarkably, π-adapted constraints need not be *Lagrangian* (in the sense of [17]) which makes the results much different from the case of mechanics (where all non-holonomic constraints are Lagrangian). π-adapted constraints include, among others, constraints important from the geometric point of view: *holonomic* constraints (i.e., constraints defined as a fibred submanifold in Y), and *constraints modelled by a distribution* or *codistribution on Y* (in particular, *semi-holonomic* constraints) [17].

In this paper, we study in detail geometric properties of π-adapted constraints and the corresponding constrained systems. We show that constraints of this kind have a fundamental geometric property (similar to non-holonomic constraints in mechanics): the constraint manifold Q carries a distribution (a subdistribution of the induced on Q-contact distribution) called *canonical distribution*. In general, it need not be completely integrable, and need not be projectable onto a distribution on the total space Y. With the help of the arising *constraint ideal* (i.e., the ideal in the algebra of differential forms generated by the annihilator of the canonical distribution) we introduce the concept of a *constrained Lagrangian system*, defined as a class of differential forms on the constraint submanifold Q. We derive the corresponding *constrained Euler–Lagrange equations*, and develop a *Hamilton–De Donder theory* for π-adapted constrained systems. We obtain a *regularity condition* which guarantees that the non-holonomic Euler–Lagrange and the non-holonomic Hamilton–De Donder equations are equivalent. For regular constrained Lagrangian systems we then construct an appropriate *Legendre transformation* as a coordinate transformation on the submanifold Q, and find explicit formulae for *constrained momenta*. Similarly as in non-holonomic mechanics, it turns out that a Hamiltonian is rather a *class of differential n-forms* which contains a *closed* form if the constraint ideal is closed, i.e., the constraints are *semi-holonomic*). Then the Hamiltonian locally arises from a *Hamilton function*. Our approach closely follows a geometric formulation of non-holonomic mechanics and field theory in jet bundles, introduced by the first of us (see [13, 14, 15, 17]), and our previous work on constrained Hamiltonian mechanics [29].

2. Lagrangian systems on fibred manifolds

In what follows, we shall use standard concepts from the theory of jet bundles and the calculus of variations on fibred manifolds. For more details, we refer to [9, 10] or [5, 12, 26].

All manifolds and mappings throughout the paper are assumed to be smooth. Summation over repeated indices is always understood, unless otherwise explicitly stated.

Let $\pi : Y \to X$ be a fibred manifold, $\dim X = n$, $\dim Y = m + n$. Consider its jet prolongations $\pi_1 : J^1Y \to X$, $\pi_2 : J^2Y \to X$, and natural projections $\pi_{1,0} : J^1Y \to Y$, $\pi_{2,1} : J^2Y \to J^1Y$ and $\pi_{2,0} : J^2Y \to Y$. We denote by (x^i, y^σ), where $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, local fibred coordinates on Y, and by $\left(x^i, y^\sigma, y^\sigma_j\right)$ and $\left(x^i, y^\sigma, y^\sigma_j, y^\sigma_{jk}\right)$, where $1 \leqslant j \leqslant k \leqslant n$, the associated coordinates on J^1Y and J^2Y, respectively. We put

$$\omega_0 = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \qquad \omega_j = i_{\partial/\partial x^j}\omega_0. \tag{2.1}$$

A mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, is called a *section* of π if $\pi \circ \gamma = \mathrm{id}_U$. We denote by $J^1\gamma$ and $J^2\gamma$ the first and the second jet prolongation of γ, respectively. Note that $J^1\gamma$ (respectively $J^2\gamma$) is a section of π_1 (respectively π_2). A section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

A vector field ξ on Y is called π*-vertical* if $T\pi \cdot \xi = 0$, and π*-projectable* if $T\pi \cdot \xi = \xi_0 \circ \pi$ for a vector field ξ_0 on X. Analogous definitions apply for vector fields on J^1Y and J^2Y with respect to different projections. A q-form η on J^1Y is called π_1*-horizontal* (respectively $\pi_{1,0}$*-horizontal*) if $i_\xi\eta = 0$ for every π_1-vertical (respectively $\pi_{1,0}$-vertical) vector field ξ on J^1Y. η is called *contact* if $J^1\gamma^*\eta = 0$ for every section γ of π. A contact form η is called *1-contact* if for every vertical vector field ξ the form $i_\xi\eta$ is π_1-horizontal; it is called *k-contact*, where $2 \leqslant k \leqslant q$, if for every vertical vector field ξ the form $i_\xi\eta$ is $(k-1)$-contact. We denote by $\Omega^{p-q,q}(J^1Y)$ the module of q-contact p-forms on J^1Y, and by $\Omega^{p-q,q}_Y(J^1Y)$ its submodule consisting of $\pi_{1,0}$-horizontal forms.

Next we denote by h, p and p_k $(k \geqslant 1)$ the horizontalization, contactization, and k-contactization operator, respectively. It is to be stressed that *every q-form* η *on* J^1Y *admits a unique decomposition into a sum of a horizontal and k-contact forms,* $1 \leqslant k \leqslant q$ (called the *horizontal, 1-contact, ..., q-contact component of* η), as follows [10]:

$$\pi^*_{2,1}\eta = h\eta + p_1\eta + p_2\eta + \cdots + p_q\eta. \tag{2.2}$$

To simplify calculations, it is convenient instead of a canonical basis of 1-forms, i.e. $\left(\mathrm{d}x^i, \mathrm{d}y^\sigma, \mathrm{d}y^\sigma_j\right)$ on J^1Y and $\left(\mathrm{d}x^i, \mathrm{d}y^\sigma, \mathrm{d}y^\sigma_j, \mathrm{d}y^\sigma_{jl}\right)$ on J^2Y, to use a basis adapted to the contact structure, i.e. $\left(\mathrm{d}x^i, \omega^\sigma, \mathrm{d}y^\sigma_j\right)$ and $\left(\mathrm{d}x^i, \omega^\sigma, \omega^\sigma_j, \mathrm{d}y^\sigma_{jl}\right)$, respectively, where

$$\omega^\sigma = \mathrm{d}y^\sigma - y^\sigma_i\,\mathrm{d}x^i, \qquad \omega^\sigma_j = \mathrm{d}y^\sigma - y^\sigma_{ji}\,\mathrm{d}x^i \tag{2.3}$$

are local canonical contact 1-forms. In such a basis, elements of the module $\Omega^{p-q,q}_Y(J^1Y)$ where $q \geqslant 1$, are expressed by means of wedge products containing exactly q of the forms ω^σ and $p - q\ \mathrm{d}x^i$.

If f is a function on J^1Y, we have by (2.2) the exterior derivative $\mathrm{d}f$ canonically split into the horizontal and contact component,

$$\pi_{2,1}^* \,\mathrm{d}f = \mathrm{d}(f \circ \pi_{2,1}) = h\,\mathrm{d}f + p\,\mathrm{d}f, \tag{2.4}$$

with

$$h\,\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}x^j}\,\mathrm{d}x^j, \tag{2.5}$$

where $\mathrm{d}/\mathrm{d}x^j$, $1 \leqslant j \leqslant n$, denotes the jth total derivative operator (also called the jth formal derivative operator),

$$\frac{\mathrm{d}}{\mathrm{d}x^j} = \frac{\partial}{\partial x^j} + y_j^\sigma \frac{\partial}{\partial y^\sigma} + y_{ij}^\sigma \frac{\partial}{\partial y_i^\sigma}. \tag{2.6}$$

For convenience of notation we also use the 'cut' total derivative operators,

$$\frac{\mathrm{d}'}{\mathrm{d}x^j} = \frac{\partial}{\partial x^j} + y_j^\sigma \frac{\partial}{\partial y^\sigma} = \frac{\mathrm{d}}{\mathrm{d}x^j} - y_{ij}^\sigma \frac{\partial}{\partial y_i^\sigma}. \tag{2.7}$$

By a *first-order Lagrangian* we shall mean a horizontal n-form on J^1Y. With a Lagrangian λ there is associated a unique at most 2-contact n-form Θ_λ such that $p_1\,\mathrm{d}\Theta_\lambda$ is $\pi_{1,0}$-horizontal. The n-form Θ_λ is called the *Poincaré–Cartan form*, and the $(n+1)$-form

$$E_\lambda = p_1\,\mathrm{d}\Theta_\lambda \tag{2.8}$$

is called the *Euler–Lagrange form* of the Lagrangian λ [9]. In fibred coordinates where

$$\lambda = L\omega_0, \tag{2.9}$$

we have

$$\Theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y_j^\sigma}\omega^\sigma \wedge \omega_j, \tag{2.10}$$

and

$$E_\lambda = E_\sigma \omega^\sigma \wedge \omega_0, \qquad \text{where} \quad E_\sigma = \frac{\partial L}{\partial y^\sigma} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial y_j^\sigma}. \tag{2.11}$$

We write

$$E_\sigma = A_\sigma + B_{\sigma\nu}^{ji} y_{ij}^\nu, \tag{2.12}$$

where obviously

$$B_{\sigma\nu}^{ji} = -\frac{\partial^2 L}{\partial y_i^\nu \partial y_j^\sigma}, \qquad A_\sigma = \frac{\partial L}{\partial y^\sigma} - \frac{\partial^2 L}{\partial x^j \partial y_j^\sigma} - \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} y_j^\nu. \tag{2.13}$$

The E_σ are affine in the variables y_{ij}^ν, i.e. A_σ and $B_{\sigma\nu}^{ji}$ are functions of $\left(x^i, y^\rho, y_l^\rho\right)$ (the $B_{\sigma\nu}^{ji}$ need not be symmetric in the upper indices).

It is known that a section γ of π is an *extremal* of λ if it satisfies the equation

$$J^1\gamma^* i_\xi\,\mathrm{d}\Theta_\lambda = 0 \quad \text{for every } \pi_1 - \text{vertical vector field } \xi \text{ on } J^1Y \tag{2.14}$$

[7, 9, 5]. This is the intrinsic form of the *Euler–Lagrange equations*; in fibred coordinates it takes the familiar form of a system of m second-order PDEs for components $\gamma^\nu = q^\nu \circ \gamma$, $1 \leqslant \nu \leqslant m$, of γ,

$$\left(\frac{\partial L}{\partial y^\sigma} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial y_j^\sigma}\right) \circ J^2\gamma = 0. \tag{2.15}$$

3. Hamilton–De Donder equations in a slightly generalized setting

Let λ be a Lagrangian on J^1Y and Θ_λ its Poincaré–Cartan n-form. Equation

$$\delta^* i_\xi \, \mathrm{d}\Theta_\lambda = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y, \tag{3.1}$$

is the well-known intrinsic form of *Hamilton–De Donder equations* [7]. Solutions of (3.1) are called *Hamilton extremals* of the Lagrangian λ; note that they are sections of the fibred manifold $\pi_1 : J^1Y \to X$. Obviously, if γ is an extremal of λ then $J^1\gamma$ is a Hamilton extremal. On the other hand, Hamilton–De Donder equations may posses solutions which are not *holonomic* sections of π_1 (such a section need not correspond to an extremal). However, if the Lagrangian satisfies the *regularity condition*

$$\det\left(\frac{\partial^2 L}{\partial y_l^\nu \partial y_j^\sigma}\right) \neq 0 \tag{3.2}$$

then every solution of the Hamilton–De Donder equations is holonomic, and, consequently, solutions of the Euler–Lagrange and Hamilton–De Donder equations of λ are in bijective correspondence. In this case, in a neighbourhood of every point in J^1Y there exists a coordinate transformation $\left(x^i, y^\sigma, y_j^\sigma\right) \to \left(x^i, y^\sigma, p_\sigma^j\right)$, called *Legendre transformation*, such that Θ_λ takes a *canonical form*

$$\Theta_\lambda = -H\omega_0 + p_\sigma^j \, \mathrm{d}y^\sigma \wedge \omega_j, \tag{3.3}$$

where

$$p_\sigma^j = \frac{\partial L}{\partial y_j^\sigma}, \qquad H = -L + p_\sigma^j y_j^\sigma. \tag{3.4}$$

In Legendre coordinates Hamilton–De Donder equations (3.1) read

$$\frac{\partial\left(p_\sigma^j \circ \delta\right)}{\partial x^j} = -\frac{\partial H}{\partial y^\sigma}, \qquad \frac{\partial(y^\sigma \circ \delta)}{\partial x^j} = \frac{\partial H}{\partial p_\sigma^j}, \tag{3.5}$$

where the functions on the right-hand side are considered along δ.

Let us return to Euler–Lagrange equations (2.14) of λ and note that they do not change if instead of the Poincaré–Cartan $(n+1)$-form $\mathrm{d}\Theta_\lambda$ one takes

$$\alpha = \mathrm{d}\Theta_\lambda + F, \tag{3.6}$$

where F is *any at least 2-contact* $(n+1)$-form on J^1Y. More generally, we have the following equivalence relation on $(n+1)$-forms on J^1Y [13]:

$$\alpha_1 \sim \alpha_2 \quad \text{if} \quad \alpha_1 - \alpha_2 \text{ is at least 2-contact.} \tag{3.7}$$

We denote by $[\alpha]$ the class of α.

The class of $\mathrm{d}\Theta_\lambda$ can be characterized as follows:

Proposition 3.1.

(1) Every at most 2-contact form $\alpha \in [d\Theta_\lambda]$ is, in fibred coordinates, expressed as follows:

$$\alpha = \mathrm{d}\Theta_\lambda + F_{\sigma\nu}^i \omega^\sigma \wedge \omega^\nu \wedge \omega_i + f_{\sigma\nu}^{ij} \omega^\sigma \wedge \mathrm{d}\omega^\nu \wedge \omega_{ij} + f_{\sigma\nu}^{ijk} \mathrm{d}\omega^\sigma \wedge \mathrm{d}\omega^\nu \wedge \omega_{ijk}. \tag{3.8}$$

(2) Let α' be such that

$$\alpha' - \mathrm{d}\Theta_\lambda \in \Omega_Y^{n-1,2}(J^1Y). \tag{3.9}$$

Then

$$\mathrm{d}\alpha' = 0 \quad \Leftrightarrow \quad \alpha' = \mathrm{d}\Theta_\lambda + F, \qquad \text{where} \quad F = 0. \tag{3.10}$$

Proof. The first part of the proposition is a direct consequence of the definition. To prove (2), it is enough to show that if $F \in \Omega_Y^{n-1,2}(J^1Y)$ satisfies $\mathrm{d}F = 0$ then $F = 0$. Denote $F = F^i_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_i$, where the components $F^i_{\sigma\nu}$ are skew-symmetric in the lower indices. Computing $\mathrm{d}F$ we obtain

$$\mathrm{d}F = \frac{\mathrm{d}' F^i_{\sigma\nu}}{\mathrm{d}x^i}\omega^\sigma \wedge \omega^\nu \wedge \omega_0 + \frac{\partial F^i_{\sigma\nu}}{\partial y^\rho}\omega^\rho \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_i + \frac{\partial F^i_{\sigma\nu}}{\partial y^\rho_j}\,\mathrm{d}y^\rho_j \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_i + 2F^i_{\sigma\nu}\omega^\sigma \wedge \mathrm{d}y^\nu_i \wedge \omega_0. \tag{3.11}$$

Now, we can see that $\mathrm{d}F = 0$ means that $F^i_{\sigma\nu} = 0$, i.e., $F = 0$. □

By the above proposition, the class of forms

$$\alpha' = \mathrm{d}\Theta_\lambda + F \qquad \text{where} \quad F \text{ is 2-contact and } \pi_{1,0}\text{-horizontal} \tag{3.12}$$

contains a *unique closed* representative (the form $\mathrm{d}\Theta_\lambda$). In what follows, the class (3.12) will be denoted by $[\mathrm{d}\Theta_\lambda]_Y$.

Definition 3.2. *We call the class* $[\mathrm{d}\Theta_\lambda]$ *the* Lagrangian system *(associated with the Lagrangian* λ*), and its subclass* $[\mathrm{d}\Theta_\lambda]_Y$ *the* Hamilton–De Donder system *of* λ.

In keeping with [16, 17], if $\alpha \in [\mathrm{d}\Theta_\lambda]$, we can consider the ideal $\mathcal{H}_\alpha$ in the exterior algebra on J^1Y, generated by n-forms

$$i_\xi\alpha, \qquad \text{where} \quad \xi \text{ runs over all } \pi_1\text{-vertical vector fields on } J^1Y. \tag{3.13}$$

$\mathcal{H}_\alpha$ is called the *Hamiltonian ideal* of α. Its integral sections are called *Hamilton extremals* of the $(n+1)$-form α.

Note the following:

(1) Equations for integral sections of the ideal $\mathcal{H}_{\mathrm{d}\Theta_\lambda}$ are Hamilton–De Donder equations of λ.
(2) *Euler–Lagrange equations* (2.14) of λ can be interpreted as equations for *holonomic* integral sections of the Hamiltonian ideal $\mathcal{H}_{\mathrm{d}\Theta_\lambda}$.
(3) Euler–Lagrange equations of λ are equations for holonomic integral sections of *any* Hamiltonian ideal $\mathcal{H}_\alpha$, where $\alpha \in [\mathrm{d}\Theta_\lambda]$.
(4) Considering different elements α in the class (3.12) provides *different* equations for Hamilton extremals (called *Hamilton equations of* α associated with λ).

Hamilton equations associated with a general (closed) $(n+1)$-form $\alpha \in [\mathrm{d}\Theta_\lambda]$ are studied in [16]. A key concept in Hamiltonian theory is that of *regularity*. For Hamilton–De Donder systems the geometric meaning of regularity can be expressed as follows (cf [11, 16, 17]):

Definition 3.3. $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$ *is called* regular *if a system of generators of* $\mathcal{H}_\alpha$ *has the maximal rank (i.e. equal to* $m + mn$*). A Lagrangian* λ *on* J^1Y *is called* (De Donder) regular *if in the class* $[\mathrm{d}\Theta_\lambda]_Y$ *there exists a regular representative.*

Theorem 3.4. λ *is (De Donder) regular if and only if*

$$\det\left(\frac{\partial^2 L}{\partial y^\nu_k \partial y^\sigma_j}\right) \neq 0 \qquad \text{i.e.} \quad \det\left(B^{jk}_{\sigma\nu}\right) \neq 0. \tag{3.14}$$

Moreover, if λ *is regular then every form* $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$ *is regular. Consequently,*

(1) every Hamilton extremal of α *is holonomic,*
(2) Hamilton equations of α *are equivalent to the Euler–Lagrange equations of* λ*,*
(3) Hamilton equations of all α *(though different) are equivalent, i.e., have the same solutions,*
(4) every Hamilton extremal of α *is a prolongation of an extremal of* λ*.*

Proof. Computing (3.13) explicitly we obtain that for $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$, $\mathcal{H}_\alpha$ can be generated by the following system of $m + mn$ differential n-forms,

$$A_\sigma \omega_0 + \left(2F^j_{\sigma\nu} + \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_j} - \frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_j}\right)\omega^\nu \wedge \omega_j + B^{ij}_{\sigma\nu}\, \mathrm{d}y^\nu_j \wedge \omega_i, \qquad B^{ij}_{\sigma\nu}\omega^\nu \wedge \omega_j, \tag{3.15}$$

where $1 \leqslant \sigma \leqslant m$, $1 \leqslant i \leqslant n$, and the A_σ and $B^{ij}_{\sigma\nu}$ are given by (2.13). This means that the matrix of generators of $\mathcal{H}_\alpha$ is the following matrix with $m + mn$ rows and $1 + mn + mn^2$ columns:

$$\begin{pmatrix} A_\sigma & 2F^j_{\sigma\nu} + \dfrac{\partial^2 L}{\partial y^\sigma \partial y^\nu_j} - \dfrac{\partial^2 L}{\partial y^\nu \partial y^\sigma_j} & B^{ij}_{\sigma\nu} \\ 0 & B^{ij}_{\sigma\nu} & 0 \end{pmatrix}. \tag{3.16}$$

If $\mathcal{H}_\alpha$ is regular, i.e., the above matrix has the maximal rank, then the square matrix $\left(B^{ij}_{\sigma\nu}\right)$ is regular. Conversely, if $\left(B^{ij}_{\sigma\nu}\right)$ is regular then the rank of (3.16) is equal to $m + mn$. Indeed, since all rows of $\left(B^{ij}_{\sigma\nu}\right)$ (labelled by (σ, i)) are linearly independent, for every fixed i, the matrix $\left(B^{ij}_{\sigma\nu}\right)$ with m rows labelled by σ, and mn columns labelled by (ν, j), has the maximal rank, m. Consequently, the matrix $\left(B^{ij}_{\sigma\nu}\right)$ with m rows labelled by σ and mn^2 columns labelled by (i, ν, j), appearing in the right upper corner of (3.16), has rank m. This proves that the corresponding form α is regular, i.e., λ is regular. Moreover, we can see that regularity does not depend on the choice of functions $F^j_{\sigma\nu}$, i.e., of $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$.

The remaining parts of theorem 3.4 now follow easily. From the generators (3.15) of $\mathcal{H}_\alpha$ we can see that if the matrix $\left(B^{ij}_{\sigma\nu}\right)$ is regular then $\delta^*\left(B^{ij}_{\sigma\nu}\omega^\nu \wedge \omega_j\right) = 0$ means that $\delta^*(\omega^\nu \wedge \omega_j) = 0$ for all ν, j, i.e.,

$$0 = \left(\mathrm{d}(y^\nu \circ \delta) - \left(y^\nu_i \circ \delta\right)\mathrm{d}x^i\right) \wedge \omega_j = \left(\frac{\partial(y^\nu \circ \delta)}{\partial x^i}\,\mathrm{d}x^i - \left(y^\nu_i \circ \delta\right)\mathrm{d}x^i\right) \wedge \omega_j$$
$$= \left(\frac{\partial(y^\nu \circ \delta)}{\partial x^j} - \left(y^\nu_j \circ \delta\right)\right)\omega_0 \quad \Leftrightarrow \quad y^\nu_j \circ \delta = \frac{\partial(y^\nu \circ \delta)}{\partial x^j}. \tag{3.17}$$

Hence, every solution of $\mathcal{H}_\alpha$ is holonomic, proving (1).

If δ is a solution of Hamilton equations of α then by (1), $\delta = J^1\gamma$ for a section γ of π. Hence, for every π_1-vertical vector field ξ, $0 = \delta^* i_\xi \alpha = J^1\gamma^* i_\xi \alpha = J^1\gamma^* i_\xi\, \mathrm{d}\Theta_\lambda$, i.e., γ is an extremal of λ, and we get a bijective correspondence between solutions of the Euler–Lagrange equations and *any* associated Hamilton equations of $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$. This means that assertions (2), (3) and (4) are true. □

By (3.3), (3.4) we get that every $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$ has a local *canonical form*

$$\alpha = -\mathrm{d}H \wedge \omega_0 + \mathrm{d}p^j_\sigma \wedge \mathrm{d}y^\sigma \wedge \omega_j + F, \tag{3.18}$$

where $F \in \Omega^{n-1,2}_Y(J^1Y)$. Moreover, if λ is regular then the momenta p^j_σ are independent, and $\left(x^i, y^\sigma, p^j_\sigma\right)$ are local coordinates on J^1Y. In these coordinates, generators of $\mathcal{H}_\alpha$ take the form

$$
\begin{aligned}
&-\left(\frac{\partial H}{\partial y^\sigma}+2F^j_{\sigma\nu}y^\nu_j\right)\omega_0+2F^j_{\sigma\nu}\,\mathrm{d}y^\nu\wedge\omega_j-\mathrm{d}p^j_\sigma\wedge\omega_j,\\
&-\frac{\partial H}{\partial p^j_\sigma}\omega_0+\mathrm{d}y^\sigma\wedge\omega_j,
\end{aligned}
\tag{3.19}
$$

and Hamilton equations of α read

$$
\frac{\partial\left(p^j_\sigma\circ\delta\right)}{\partial x^j}=-\frac{\partial H}{\partial y^\sigma}+2F^j_{\sigma\nu}\left(\frac{\partial(y^\nu\circ\delta)}{\partial x^j}-\left(y^\nu_j\circ\delta\right)\right),\qquad \frac{\partial(y^\sigma\circ\delta)}{\partial x^j}=\frac{\partial H}{\partial p^j_\sigma},
\tag{3.20}
$$

(where the functions on the right-hand side are considered along δ). Since by (3.4) $\partial H/\partial p^j_\sigma = y^\sigma_j$, equations (3.20) are apparently equivalent to Hamilton–De Donder equations (3.5).

Summarizing, we can see that for regular Lagrangians, Hamilton–De Donder equations are obtained from any $(n+1)$-form $\alpha \in [\mathrm{d}\Theta_\lambda]_Y$.

4. Non-holonomic constraints

A *non-holonomic constraint* in J^1Y is defined to be a fibred submanifold Q of $\pi_{1,0}$, codim $Q = K$, where $1 \leqslant K \leqslant mn-1$. Denote by $\iota : Q \to J^1Y$ the canonical embedding of the submanifold Q into J^1Y.

Throughout this paper, we shall consider a class of non-holonomic constraints in J^1Y, characterized as follows:

Definition 4.1. *A non-holonomic constraint $Q \subset J^1Y$ is called* π-adapted *(of rank κ) if it can be locally defined by a system of kn first-order partial differential equations in normal form,*

$$
f^a_j \equiv y^{m-k+a}_j - g^a_j\left(x^i, y^\sigma, y^s_l\right) = 0,\qquad 1\leqslant a\leqslant k<m,\qquad 1\leqslant j\leqslant n,
\tag{4.1}
$$

such that

$$
\operatorname{rank}\left(\frac{\partial f^a_j}{\partial y^\sigma_i}\right)=\kappa<m,\qquad \textit{where}\quad (a,j,i)\ \textit{label rows and}\ \sigma\ \textit{label columns}.
\tag{4.2}
$$

Remark 4.2.

(1) Functions g^a_j above depend on $x^i, 1\leqslant i\leqslant n, y^\sigma, 1\leqslant\sigma\leqslant m$, and $y^s_l, 1\leqslant s\leqslant m-k, 1\leqslant l\leqslant n$.

(2) corank $Q = kn$.

(3) For all $a, b = 1, 2, \dots, k$, $s = 1, 2, \dots, m-k$, and $i, j = 1, 2, \dots, n$

$$
\frac{\partial f^a_j}{\partial y^{m-k+b}_i}=\delta^a_b\delta^i_j,\qquad \frac{\partial f^a_j}{\partial y^s_i}=-\frac{\partial g^a_j}{\partial y^s_i}.
\tag{4.3}
$$

Taking into account rank condition (4.2) we can see that $\kappa \geqslant k$.

(4) From (4.1) one can see that

$$
\operatorname{rank}\left(\frac{\partial f^a_j}{\partial y^\sigma_i}\right)=\operatorname{rank}\left(-\frac{\partial g^a_j}{\partial y^s_i}\ \delta^a_b\delta^i_j\right)=\max=kn,
\tag{4.4}
$$

where (a, j) label rows and $(\sigma, i) = (s, b, i)$ label columns, the matrix in (4.4) being a $(kn \times mn)$ matrix with the $(kn \times kn)$ unit submatrix. This means that, indeed, Q is a fibred submanifold of $\pi_{1,0}$.

Definition 4.3. *Let (V, ψ) be a fibred chart on Y, (V_1, ψ_1) the associated chart on J^1Y. Let $U \subset V_1$ be an open set. On U consider the following 1-forms,*

$$\phi_j^{ai} = f_j^a \,\mathrm{d}x^i + \frac{1}{n}\frac{\partial f_j^a}{\partial y_i^\sigma}\omega^\sigma = \left(y_j^{m-k+a} - g_j^a\right)\mathrm{d}x^i - \frac{1}{n}\left(\frac{\partial g_j^a}{\partial y_i^s}\omega^s - \delta_j^i \omega^{m-k+a}\right),$$
$$1 \leqslant a \leqslant k, \quad 1 \leqslant i, j \leqslant n, \tag{4.5}$$

and set

$$\tilde{\mathcal{C}}_U = \mathrm{annih}\left\{\phi_j^{ai}\right\}, \qquad \mathcal{C}_U = \mathrm{annih}\left\{\phi_j^{ai}, \mathrm{d}f_j^a\right\}. \tag{4.6}$$

The distribution $\tilde{\mathcal{C}}_U$ and $\mathcal{C}_U$ on U will be called extended local constraint distribution *and* local constraint distribution *associated with the constraint Q, respectively.*

Apparently, $\mathcal{C}_U$ is a subdistribution of $\tilde{\mathcal{C}}_U$. Note that we have another distinguished subdistribution of $\tilde{\mathcal{C}}_U$, of constant corank k, annihilated by the following system of linearly independent 1-forms on U

$$\begin{aligned}\phi^a &= \phi_j^{ai}\delta_i^j = f_i^a \,\mathrm{d}x^i + \frac{1}{n}\frac{\partial f_i^a}{\partial y_i^\sigma}\omega^\sigma \\ &= \left(y_i^{m-k+a} - g_i^a\right)\mathrm{d}x^i - \frac{1}{n}\frac{\partial g_i^a}{\partial y_i^s}\omega^s + \omega^{m-k+a}, \qquad 1 \leqslant a \leqslant k.\end{aligned} \tag{4.7}$$

In what follows, we shall use the following notation,

$$\begin{aligned}\bar{\omega}^\sigma &= \iota^*\omega^\sigma, \\ \varphi_j^{ai} &= \iota^*\phi_j^{ai} = \frac{1}{n}\left(\frac{\partial f_j^a}{\partial y_i^\sigma}\circ\iota\right)\bar{\omega}^\sigma = -\frac{1}{n}\frac{\partial g_j^a}{\partial y_i^s}\omega^s + \frac{1}{n}\delta_j^i\bar{\omega}^{m-k+a}, \\ \varphi^a &= \iota^*\phi^a = \varphi_j^{ai}\delta_i^j = \frac{1}{n}\left(\frac{\partial f_i^a}{\partial y_i^\sigma}\circ\iota\right)\bar{\omega}^\sigma \\ &= -\frac{1}{n}\frac{\partial g_i^a}{\partial y_i^s}\omega^s + \bar{\omega}^{m-k+a} = -g_i^a\,\mathrm{d}x^i - \frac{1}{n}\frac{\partial g_i^a}{\partial y_i^s}\omega^s + \mathrm{d}y^{m-k+a},\end{aligned} \tag{4.8}$$

where $1 \leqslant \sigma \leqslant m$, $1 \leqslant a \leqslant k$, $1 \leqslant i, j \leqslant n$, and we have used that $\bar{\omega}^s = \omega^s$, $1 \leqslant s \leqslant m-k$.

Proposition 4.4. *At the points of $Q \cap U$, $\mathcal{C}_U$ is a distribution of corank κ on $Q \cap U$, annihilated by the forms φ_j^{ai}.*

Proof. $\mathcal{C}_U$ is a subdistribution of the distribution $\mathcal{Q}_U$ on U, annihilated by the (kn independent) 1-forms $\mathrm{d}f_j^a$. However, $\mathcal{Q}_U$ has the constraint submanifold Q (precisely $Q \cap U$) as one of the integral submanifolds. This means that along $Q \cap U$ the vector fields belonging to $\mathcal{C}_U$ are *tangent* to $Q \cap U$, and are annihilated by the 1-forms $\iota^*\phi_j^{ai} = \varphi_j^{ai}$. □

Now, we shall show that the system of *local* constraint distributions along the constraint submanifold Q unites into a *global distribution on Q*.

Theorem 4.5. *Let $\iota : Q \to J^1Y$ be the canonical embedding of the submanifold Q into J^1Y. Then local 1-forms $\varphi_j^{ai} = \iota^*\phi_j^{ai}$, $1 \leqslant a \leqslant k$, $1 \leqslant i, j \leqslant n$, annihilate a distribution of corank κ on Q, i.e., a subbundle of the tangent bundle $TQ \to Q$ of corank κ.*

Proof. Let $\mathcal{C}_{U_1}$, $\mathcal{C}_{U_2}$ be two local constraint distributions defined on open sets U_1, U_2 such that $U_1 \cap U_2 \cap Q \neq \emptyset$. Denote by $\left(x^i, y^\sigma, y_j^\sigma\right)$ and $\left(x'^i, y'^\sigma, y_j'^\sigma\right)$ the associated fibred coordinates on U_1 and U_2, respectively. If $f^a_j = 0$ and $f'^a_j = 0$ are equations of the constraint Q on U_1 and U_2, respectively, we have

$$\begin{aligned} \mathcal{C}_{U_1} &= \text{annih}\left\{\phi^{ai}_j = f^a_j\,\text{d}x^i + \frac{1}{n}\frac{\partial f^a_j}{\partial y^\sigma_i}\omega^\sigma, \text{d}f^a_j\right\}, \\ \mathcal{C}_{U_2} &= \text{annih}\left\{\phi'^{ai}_j = f'^a_j\,\text{d}x'^i + \frac{1}{n}\frac{\partial f'^a_j}{\partial y'^\sigma_i}\omega'^\sigma, \text{d}f'^a_j\right\}, \end{aligned} \tag{4.9}$$

and for some functions c^{al}_{jb} on $U_1 \cap U_2 \cap Q$,

$$\text{d}f'^a_j(x) = c^{al}_{jb}(x)\,\text{d}f^b_l(x) \tag{4.10}$$

at each point $x \in U_1 \cap U_2 \cap Q$. The latter relation means that at these points,

$$\frac{\partial f'^a_j}{\partial y'^\sigma_i} = c^{al}_{jb}\frac{\partial f^b_l}{\partial y^\nu_p}\frac{\partial y^\nu_p}{\partial y'^\sigma_i} = c^{al}_{jb}\frac{\partial f^b_l}{\partial y^\nu_p}\frac{\partial y^\nu}{\partial y'^\sigma}\frac{\partial x'^i}{\partial x^p}. \tag{4.11}$$

Now,

$$\begin{aligned} n\varphi'^{ai}_j &= n\iota^*\phi'^{ai}_j = \left(\frac{\partial f'^a_j}{\partial y'^\sigma_i}\circ\iota\right)\bar{\omega}'^\sigma = c^{al}_{jb}\left(\left(\frac{\partial f^b_l}{\partial y^\nu_p}\circ\iota\right)\frac{\partial y^\nu}{\partial y'^\sigma}\frac{\partial x'^i}{\partial x^p}\frac{\partial y'^\sigma}{\partial y^\rho}\right)\bar{\omega}^\rho \\ &= c^{al}_{jb}\frac{\partial x'^i}{\partial x^p}\left(\frac{\partial f^b_l}{\partial y^\nu_p}\circ\iota\right)\bar{\omega}^\nu = \hat{c}^{ali}_{jbp}\varphi^{bp}_l \end{aligned} \tag{4.12}$$

(with an obvious notation for $\hat{c}^{ali}_{jbp}$), meaning that on $U_1 \cap U_2 \cap Q$ the 1-forms φ^{ai}_j and φ'^{ai}_j annihilate the same distribution. □

Definition 4.6 ([17]). *The distribution*

$$\mathcal{C} = \text{annih}\{\varphi^{ai}_j, 1 \leqslant a \leqslant k, 1 \leqslant i, j \leqslant n\}, \tag{4.13}$$

on Q, defined in theorem 4.5 is called a canonical distribution *of the constraint Q, and the 1-forms φ^{ai}_j are called* canonical constraint 1-forms. *The ideal $\mathcal{I}$ in the exterior algebra of differential forms on Q generated by canonical constraint 1-forms is called the* constraint ideal. *A pair $(Q, \mathcal{I})$ where Q is a constraint in J^1Y and $\mathcal{I}$ is its constraint ideal is called a* constraint structure *on π_1.*

Remark 4.7. Due to the rank condition (4.2), in a neighbourhood of every point in Q there exists a system of κ linearly independent annihilating 1-forms for $\mathcal{C}$. Moreover, κ of the contact forms $\bar{\omega}^\sigma$ can be expressed by means of these constraint forms and the remaining 'omegas'. Without loss of generality, we may assume that

$$\mathcal{C} = \text{annih}\{\varphi^\alpha, 1 \leqslant \alpha \leqslant \kappa\}, \tag{4.14}$$

where

$$\varphi^\alpha = \bar{\omega}^{m-\kappa+\alpha} - \sum_{r=1}^{m-\kappa} G^\alpha_r\omega^r, \qquad 1 \leqslant \alpha \leqslant \kappa, \tag{4.15}$$

for appropriate functions G^α_s. We also have

$$\mathcal{C} = \text{annih}\{\varphi^a, \varphi^\alpha, 1 \leqslant a \leqslant k, 1 \leqslant \alpha \leqslant \kappa - k\}, \tag{4.16}$$

where φ^a are defined in (4.8) and φ^α are the forms above.

Proposition 4.8. *The canonical distribution $\mathcal{C}$ is locally spanned by the following independent vector fields,*

$$\frac{\partial_c}{\partial x^l} = \frac{\partial}{\partial x^l} + \sum_{\alpha=1}^{\kappa-k}\left(y_l^{m-\kappa+\alpha} - \sum_{r=1}^{m-\kappa} G_r^\alpha y_l^r\right)\frac{\partial}{\partial y^{m-\kappa+\alpha}} + \left(g_l^a - \sum_{r=1}^{m-\kappa}\Gamma_r^a y_l^r\right)\frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leqslant l \leqslant n,$$
$$\frac{\partial_c}{\partial y^r} = \frac{\partial}{\partial y^r} + \sum_{\alpha=1}^{\kappa-k} G_r^\alpha \frac{\partial}{\partial y^{m-\kappa+\alpha}} + \sum_{a=1}^{k}\Gamma_r^a\frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leqslant r \leqslant m-\kappa, \tag{4.17}$$
$$\frac{\partial}{\partial y_j^s}, \qquad 1 \leqslant s \leqslant m-k, \quad 1 \leqslant j \leqslant n,$$

where

$$\Gamma_r^a = \frac{1}{n}\left(\frac{\partial g_i^a}{\partial y_i^r} + \sum_{\alpha=1}^{\kappa-k}\frac{\partial g_i^a}{\partial y_i^{m-\kappa+\alpha}} G_r^\alpha\right), \qquad 1 \leqslant a \leqslant k, \quad 1 \leqslant r \leqslant m-\kappa, \tag{4.18}$$

or, equivalently, by

$$\frac{\mathrm{d}_c'}{\mathrm{d}x^l} = \frac{\partial_c}{\partial x^l} + \sum_{r=1}^{m-\kappa} y_l^r\frac{\partial_c}{\partial y^r} = \frac{\partial}{\partial x^l} + \sum_{s=1}^{m-k} y_l^s\frac{\partial}{\partial y^s} + g_l^a\frac{\partial}{\partial y^{m-k+a}} = \frac{\mathrm{d}'}{\mathrm{d}x^l}\circ\iota, \qquad 1 \leqslant l \leqslant n,$$
$$\frac{\partial_c}{\partial y^r}, \qquad 1 \leqslant r \leqslant m-\kappa, \tag{4.19}$$
$$\frac{\partial}{\partial y_j^s}, \qquad 1 \leqslant s \leqslant m-k, 1 \leqslant j \leqslant n.$$

Proof. A vector field ξ on Q,

$$\xi = \xi^l\frac{\partial}{\partial x^l} + \Xi^\sigma\frac{\partial}{\partial y^\sigma} + \Xi_j^s\frac{\partial}{\partial y_j^s}, \tag{4.20}$$

(where summations run over $l = 1, \ldots, n, \sigma = 1, \ldots, m$, and $s = 1, \ldots, m-k$) belongs to the canonical distribution $\mathcal{C}$ iff for all $a = 1, \ldots, k$, and $\alpha = 1, \ldots, \kappa-k$,

$$\begin{aligned} i_\xi\varphi^a &= -\frac{1}{n}\sum_{s=1}^{m-k}\frac{\partial g_i^a}{\partial y_i^s}\left(\Xi^s - y_l^s\xi^l\right) + \Xi^{m-k+a} - g_l^a\xi^l = 0, \\ i_\xi\varphi^\alpha &= \Xi^{m-\kappa+\alpha} - y_l^{m-\kappa+\alpha}\xi^l - \sum_{r=1}^{m-\kappa} G_r^\alpha\left(\Xi^r - y_l^r\xi^l\right) = 0. \end{aligned} \tag{4.21}$$

These conditions give us

$$\begin{aligned} \Xi^{m-k+a} &= \frac{1}{n}\sum_{s=1}^{m-k}\frac{\partial g_i^a}{\partial y_i^s}\Xi^s + \left(g_l^a - \frac{1}{n}\sum_{s=1}^{m-k}\frac{\partial g_i^a}{\partial y_i^s}y_l^s\right)\xi^l, \qquad 1 \leqslant a \leqslant k, \\ \Xi^{m-\kappa+\alpha} &= \sum_{r=1}^{m-\kappa} G_r^\alpha\Xi^r + \left(y_l^{m-\kappa+\alpha} - \sum_{r=1}^{m-\kappa} G_r^\alpha y_l^r\right)\xi^l, \qquad 1 \leqslant \alpha \leqslant \kappa-k. \end{aligned} \tag{4.22}$$

Hence,

$$\begin{aligned} \Xi^{m-k+a} &= \frac{1}{n}\sum_{r=1}^{m-\kappa}\left(\frac{\partial g_i^a}{\partial y_i^r} + \sum_{\alpha=1}^{\kappa-k}\frac{\partial g_i^a}{\partial y_i^{m-\kappa+\alpha}}G_r^\alpha\right)\Xi^r \\ &\quad + \left(g_l^a - \frac{1}{n}\sum_{r=1}^{m-\kappa}\left(\frac{\partial g_i^a}{\partial y_i^r} + \sum_{\alpha=1}^{\kappa-k}\frac{\partial g_i^a}{\partial y_i^{m-\kappa+\alpha}}G_r^\alpha\right)y_l^r\right)\xi^l, \qquad 1 \leqslant a \leqslant k, \end{aligned} \tag{4.23}$$

and we get that a vector field ξ on Q belongs to $\mathcal{C}$ iff (in notation of (4.18))

$$\xi = \xi^l \frac{\partial}{\partial x^l} + \sum_{r=1}^{m-\kappa} \Xi^r \frac{\partial}{\partial y^r} + \sum_{\alpha=1}^{\kappa-k} \left(\sum_{r=1}^{m-\kappa} G^\alpha_r \Xi^r + \left(y_l^{m-\kappa+\alpha} - \sum_{r=1}^{m-\kappa} G^\alpha_r y^r_l \right) \xi^l \right) \frac{\partial}{\partial y^{m-\kappa+\alpha}}$$
$$+ \sum_{a=1}^{k} \left(\sum_{r=1}^{m-\kappa} \Gamma^a_r \Xi^r + \left(g^a_l - \sum_{r=1}^{m-\kappa} \Gamma^a_r y^r_l \right) \xi^l \right) \frac{\partial}{\partial y^{m-k+a}} + \sum_{s=1}^{m-k} \Xi^s_j \frac{\partial}{\partial y^s_j}, \tag{4.24}$$

where ξ^l, Ξ^r and Ξ^s_j, $1 \leqslant j, l \leqslant n$, $1 \leqslant r \leqslant m-\kappa$, $1 \leqslant s \leqslant m-k$, are arbitrary functions. This means that, indeed, (4.17) (respectively (4.19)) are generators of $\mathcal{C}$. □

Remark 4.9. We call the vector fields $\partial_c/\partial x^l$ and $\partial_c/\partial y^r (1 \leqslant l \leqslant n, 1 \leqslant r \leqslant m-\kappa)$ in (4.17) *constraint partial derivative operators, and* $\mathrm{d}'_c/\mathrm{d}x^l$ $(1 \leqslant l \leqslant n)$ *in* (4.19) *cut constraint total derivative operators*. For convenience of notation, we also introduce *constraint total derivative operators*

$$\frac{\mathrm{d}_c}{\mathrm{d}x^l} = \frac{\partial}{\partial x^l} + \sum_{s=1}^{m-k} y^s_l \frac{\partial}{\partial y^s} + g^a_l \frac{\partial}{\partial y^{m-k+a}} + \sum_{s=1}^{m-k} y^s_{jl} \frac{\partial}{\partial y^s_j}$$
$$= \frac{\partial_c}{\partial x^l} + \sum_{r=1}^{m-\kappa} y^r_l \frac{\partial_c}{\partial y^r} + \sum_{s=1}^{m-k} y^s_{jl} \frac{\partial}{\partial y^s_j} = \frac{\mathrm{d}'_c}{\mathrm{d}x^l} + \sum_{s=1}^{m-k} y^s_{jl} \frac{\partial}{\partial y^s_j}, \qquad 1 \leqslant l \leqslant n, \tag{4.25}$$

and the *constraint Euler–Lagrange operator* and *cut constraint Euler–Lagrange operator*, respectively,

$$\varepsilon_r = \frac{\partial_c}{\partial y^r} - \frac{\mathrm{d}_c}{\mathrm{d}x^j} \frac{\partial}{\partial y^r_j}, \qquad \varepsilon'_r = \frac{\partial_c}{\partial y^r} - \frac{\mathrm{d}'_c}{\mathrm{d}x^j} \frac{\partial}{\partial y^r_j}, \qquad 1 \leqslant r \leqslant m-\kappa. \tag{4.26}$$

Next, instead of a canonical basis $(\mathrm{d}x^i, \mathrm{d}y^\sigma, \mathrm{d}y^s_j)$ of 1-forms on Q, or a basis $(\mathrm{d}x^i, \bar{\omega}^\sigma, \mathrm{d}y^s_j)$ adapted to the induced contact structure, it is convenient to work *with bases adapted to the constraint structure*, where the canonical constraint 1-forms appear,

$$(\mathrm{d}x^i, \mathrm{d}y^r, \varphi^a, \varphi^\alpha, \mathrm{d}y^s_j), \qquad (\mathrm{d}x^i, \omega^r, \varphi^a, \varphi^\alpha, \mathrm{d}y^s_j), \tag{4.27}$$

where $1 \leqslant i, j \leqslant n$, $1 \leqslant r \leqslant m-\kappa$, $1 \leqslant s \leqslant m-k$, $1 \leqslant a \leqslant k$, and $1 \leqslant \alpha \leqslant \kappa - k$.

Remark 4.10. As stated above, we consider the canonical distribution $\mathcal{C}$ annihilated by the system of local 1-forms on the constraint manifold Q,

$$\varphi^\alpha = \omega^{m-\kappa+\alpha} - \sum_{r=1}^{m-\kappa} G^\alpha_r \omega^r = -y_i^{m-\kappa+\alpha}\,\mathrm{d}x^i - \sum_{r=1}^{m-\kappa} G^\alpha_r \omega^r + \mathrm{d}y^{m-\kappa+\alpha}, \qquad 1 \leqslant \alpha \leqslant \kappa - k,$$
$$\varphi^a = \bar{\omega}^{m-k+a} - \sum_{s=1}^{m-k} \frac{1}{n} \frac{\partial g^a_i}{\partial y^s_i} \omega^s = -g^a_i\,\mathrm{d}x^i - \sum_{s=1}^{m-k} \frac{1}{n} \frac{\partial g^a_i}{\partial y^s_i} \omega^s + \mathrm{d}y^{m-k+a}$$
$$= \bar{\omega}^{m-k+a} - \sum_{r=1}^{m-\kappa} \Gamma^a_r \omega^r - \sum_{\alpha=1}^{\kappa-k} \frac{1}{n} \frac{\partial g^a_i}{\partial y_i^{m-\kappa+\alpha}} \varphi^\alpha, \qquad 1 \leqslant a \leqslant k, \tag{4.28}$$

where Γ^a_r are defined in (4.18).

We get the following formulae which will be used later:

$$\begin{aligned}
\mathrm{d}\varphi^\alpha = & \sum_{r=1}^{m-\kappa} \frac{\mathrm{d}'_c G^\alpha_r}{\mathrm{d}x^j}\omega^r \wedge \mathrm{d}x^j + \sum_{r=1}^{m-\kappa} G^\alpha_r \,\mathrm{d}y^r_j \wedge \mathrm{d}x^j - \mathrm{d}y_i^{m-\kappa+\alpha} \wedge \mathrm{d}x^i \\
& + \sum_{r,s=1}^{m-\kappa} \frac{\partial_c G^\alpha_r}{\partial y^s}\omega^r \wedge \omega^s + \sum_{r}^{m-\kappa}\sum_{s=1}^{m-k} \frac{\partial G^\alpha_r}{\partial y^s_j}\omega^r \wedge \mathrm{d}y^s_j + a\ constraint\ form, \qquad (4.29)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\varphi^a = & \frac{\mathrm{d}'_c g^a_i}{\mathrm{d}x^j}\,\mathrm{d}x^i \wedge \mathrm{d}x^j - \sum_{r=1}^{m-\kappa}\left(\frac{\partial_c g^a_j}{\partial y^r} - \frac{\mathrm{d}'_c \Gamma^a_r}{\mathrm{d}x^j} + \frac{1}{n}\frac{\partial g^a_i}{\partial y_i^{m-\kappa+\alpha}}\frac{\mathrm{d}'_c G^\alpha_r}{\mathrm{d}x^j}\right)\omega^r \wedge \mathrm{d}x^j \\
& + \sum_{s=1}^{m-k}\left(\frac{1}{n}\frac{\partial g^a_l}{\partial y^s_l}\delta^j_i - \frac{\partial g^a_i}{\partial y^s_j}\right)\mathrm{d}y^s_j \wedge \mathrm{d}x^i \\
& - \sum_{r,s=1}^{m-\kappa}\left(\frac{\partial_c \Gamma^a_s}{\partial y^r} - \frac{1}{n}\frac{\partial g^a_i}{\partial y_i^{m-\kappa+\alpha}}\frac{\partial_c G^\alpha_s}{\partial y^r}\right)\omega^r \wedge \omega^s \\
& + \sum_{r=1}^{m-\kappa}\sum_{s=1}^{m-k}\left(\frac{\partial \Gamma^a_r}{\partial y^s_j} - \frac{1}{n}\frac{\partial g^a_i}{\partial y_i^{m-\kappa+\alpha}}\frac{\partial G^\alpha_r}{\partial y^s_j}\right)\omega^r \wedge \mathrm{d}y^s_j + a\ constraint\ form. \qquad (4.30)
\end{aligned}$$

Definition 4.11 ([17]). *A constraint Q in J^1Y is called* Lagrangian *if for a system of constraint forms φ^A, $1 \leqslant A \leqslant \kappa$, generating the constraint ideal, the $p_1 \mathrm{d}\varphi^A$ are horizontal with respect to the projection onto Y.*

In the above definition, p_1 is the operator of constraint 1-contactization, introduced in [18], assigning to a form on Q its constraint 1-contact part, defined on $\tilde{Q}$, natural prolongation of Q, which is a submanifold in J^2Y. We note that if a system of generators satisfies the condition from definition 4.11 then the same holds for any other system generating the constraint ideal [17].

Taking into account remark 4.10 we immediately obtain

Theorem 4.12. *A π-adapted constraint Q in J^1Y is Lagrangian if and only if $\kappa = k$, and*

$$\frac{1}{n}\frac{\partial g^a_l}{\partial y^s_l}\delta^j_i - \frac{\partial g^a_i}{\partial y^s_j} = 0, \qquad (4.31)$$

or, equivalently,

$$\frac{\partial g^a_1}{\partial y^s_1} = \frac{\partial g^a_2}{\partial y^s_2} = \cdots = \frac{\partial g^a_n}{\partial y^s_n}, \qquad \frac{\partial g^a_i}{\partial y^s_j} = 0, \quad i \neq j. \qquad (4.32)$$

Conditions (4.31) (respectively (4.32)) mean that equations (4.1) of Q are separable and affine in the first derivatives, i.e. of the form

$$y_j^{m-k+a} = h^a_s(x^i, y^\sigma)y^s_j + b^a_j(x^i, y^\sigma). \qquad (4.33)$$

Theorem 4.13. *Every π-adapted constraint such that $\kappa = k$, is Lagrangian.*

Proof. Computing the matrix in (4.2) we have

$$\begin{pmatrix} \frac{\partial f_1^1}{\partial y_1^1} & \dots & \frac{\partial f_1^1}{\partial y_1^m} \\ \frac{\partial f_1^2}{\partial y_1^1} & \dots & \frac{\partial f_1^2}{\partial y_1^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1^k}{\partial y_1^1} & \dots & \frac{\partial f_1^k}{\partial y_1^m} \\ \frac{\partial f_1^1}{\partial y_2^1} & \dots & \frac{\partial f_1^1}{\partial y_2^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_j^a}{\partial y_i^1} & \dots & \frac{\partial f_j^a}{\partial y_i^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n^k}{\partial y_n^1} & \dots & \frac{\partial f_n^k}{\partial y_n^m} \end{pmatrix} \sim \begin{pmatrix} -\frac{\partial g_1^1}{\partial y_1^1} & -\frac{\partial g_1^1}{\partial y_1^2} & \dots & -\frac{\partial g_1^1}{\partial y_1^{m-k}} & 1 & 0 & \dots & 0 \\ -\frac{\partial g_1^2}{\partial y_1^1} & -\frac{\partial g_1^2}{\partial y_1^2} & \dots & -\frac{\partial g_1^2}{\partial y_1^{m-k}} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_1^k}{\partial y_1^1} & -\frac{\partial g_1^k}{\partial y_1^2} & \dots & -\frac{\partial g_1^k}{\partial y_1^{m-k}} & 0 & 0 & \dots & 1 \\ -\frac{\partial g_1^1}{\partial y_2^1} & -\frac{\partial g_1^1}{\partial y_2^2} & \dots & -\frac{\partial g_1^1}{\partial y_2^{m-k}} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_j^a}{\partial y_i^1} & -\frac{\partial g_j^a}{\partial y_i^2} & \dots & -\frac{\partial g_j^a}{\partial y_i^{m-k}} & \delta_j^i\delta_1^a & \delta_j^i\delta_2^a & \dots & \delta_j^i\delta_{m-k}^a \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_n^k}{\partial y_n^1} & -\frac{\partial g_n^k}{\partial y_n^2} & \dots & -\frac{\partial g_n^k}{\partial y_n^{m-k}} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Since the rank of this matrix is equal to k, the functions g_j^a have to satisfy (4.32). Hence Q is Lagrangian by theorem 4.12. □

Definition 4.14. *A constraint Q in J^1Y is called* semi-holonomic *if the canonical distribution $\mathcal{C}$ of Q is completely integrable.*

Formulae in remark 4.10 give us the following equivalent characterizations of semi-holonomic constraints:

Theorem 4.15. *The following conditions are equivalent:*

(1) A π-adapted constraint Q in J^1Y is semi-holonomic.
(2) The constraint ideal $\mathcal{I}$ is closed.
(3) Q satisfies $\kappa = k$ (i.e., Q is Lagrangian), and

$$\frac{\mathrm{d}_c g_i^a}{\mathrm{d}x^j} = \frac{\mathrm{d}_c g_j^a}{\mathrm{d}x^i}, \qquad \varepsilon_s\big(g_j^a\big) = 0, \tag{4.34}$$

or, equivalently,

$$\frac{\mathrm{d}'_c g_i^a}{\mathrm{d}x^j} = \frac{\mathrm{d}'_c g_j^a}{\mathrm{d}x^i}, \qquad \varepsilon'_s\big(g_j^a\big) = 0. \tag{4.35}$$

Proof. It is sufficient to note that if $\kappa = k$, formulae (4.29), (4.30) simplify to

$$\mathrm{d}\varphi^a = \frac{\mathrm{d}'_c g_i^a}{\mathrm{d}x^j}\,\mathrm{d}x^i \wedge \mathrm{d}x^j - \varepsilon'_s\big(g_j^a\big)\omega^s \wedge \mathrm{d}x^j - \frac{1}{2}\frac{\partial \varepsilon'_r\big(g_i^a\big)}{\partial y_i^s}\omega^r \wedge \omega^s + \left(\frac{\partial g_j^a}{\partial y^{m-k+b}}\,\mathrm{d}x^j + \frac{1}{n}\frac{\partial^2 g_i^a}{\partial y^{m-k+b}\partial y_i^s}\omega^s\right) \wedge \varphi^b. \tag{4.36}$$

□

Remark 4.16. Note that for a Lagrangian π-adapted constraint one has

$$\varphi_j^{ai} = \iota^*\phi_j^{ai} = 0, \quad \text{for all } i \neq j,$$
$$\varphi_i^{ai} = \iota^*\phi_i^{ai} = \frac{1}{n}\left(\bar{\omega}^{m-k+a} - \frac{\partial g_i^a}{\partial y_i^s}\omega^s\right) \textit{ no summation over } i, \quad 1 \leqslant i \leqslant n. \tag{4.37}$$

Hence, for every fixed $i = 1, \ldots, n$, the 1-forms φ_i^{ai}, $1 \leqslant a \leqslant k$, look like constraint 1-forms in mechanics (non-holonomic constraints on a fibred manifold over a one-dimensional base, the x^i-axis) (cf, e.g., [13]). Therefore one could think of Lagrangian π-adapted constraints in field theory as of a 'multi-time' non-holonomic mechanics. However, there is in no case an analogy with the constraint structure in mechanics: one should note that the corank of the canonical distribution $\mathcal{C}$ is k (since only k (not kn) of the forms $\varphi_1^{a1}, \ldots, \varphi_n^{an}$, $1 \leqslant a \leqslant k$, are independent).

5. Constrained Lagrangian systems

Let $(Q, \mathcal{I})$ be a constraint structure on π_1. Since for every q-contact form η on J^1Y $\iota^*\eta$ is a q-contact form on Q, we have the following equivalence relation on $(n+1)$-forms on Q,

$$\alpha_1 \approx \alpha_2 \quad \text{if} \quad \alpha_1 - \alpha_2 = \bar{F} + \varphi, \tag{5.1}$$

where $\bar{F}$ is an at least 2-contact $(n+1)$-form on Q, and φ is a constraint $(n+1)$-form. We denote by $[[\alpha]]$ the class of α. If $\alpha_1 \sim \alpha_2$ (in the sense of definition (3.7)) then $\iota^*\alpha_1 \approx \iota^*\alpha_2$.

Remark 5.1. In the following, we shall work with *first-order Lagrangian systems whose Euler–Lagrange equations are non-trivially of the second order*. This means that equivalently,

(1) the form $d\Theta_\lambda$ is defined on J^1Y and is not projectable onto Y,
(2) Lagrangian λ is non-affine in the first derivatives,
(3) if $\alpha \sim d\Theta_\lambda$ then

$$\alpha \sim A_\sigma\omega^\sigma \wedge \omega_0 + B_{\sigma\nu}^{ij}\omega^\sigma \wedge dy_j^\nu \wedge \omega_i, \tag{5.2}$$

where $\left(B_{\sigma\nu}^{ij}\right)$ is a non-zero matrix, and A_σ, $B_{\sigma\nu}^{ij}$ are expressed by means of the Lagrangian in (2.13).

Definition 5.2. *Let λ be a Lagrangian on J^1Y, Θ_λ its Poincaré–Cartan form. We call the equivalence class* $[[\iota^* d\Theta_\lambda]]$ the constrained system *associated with λ and the constraint Q. Every element of* $[[\iota^* d\Theta_\lambda]]$ *of the form*

$$\iota^* d\Theta_\lambda + \varphi, \quad \varphi \in \mathcal{I} \tag{5.3}$$

will be called constrained Poincaré–Cartan $(n+1)$-form of λ.

We note that a general element of the class $[[\iota^* d\Theta_\lambda]]$ is of the form

$$\bar{\alpha} = \iota^* d\Theta_\lambda + \bar{F} + \varphi, \tag{5.4}$$

where $\bar{F}$ is at least 2-contact and $\varphi \in \mathcal{I}$.

For a Lagrangian $\lambda = L\omega_0$ we set

$$\bar{L} = L \circ \iota, \qquad \bar{L}_\alpha^j = \frac{\partial L}{\partial y_j^{m-\kappa+\alpha}} \circ \iota, \quad \bar{L}_a^j = \frac{\partial L}{\partial y_j^{m-k+a}} \circ \iota, \tag{5.5}$$

where $1 \leqslant \alpha \leqslant \kappa - k$, $1 \leqslant a \leqslant k$, and

$$\Theta_{\iota^*\lambda} = \bar{L}\omega_0 + \sum_{r=1}^{m-\kappa} \frac{\partial \bar{L}}{\partial y_j^r}\omega^r \wedge \omega_j. \tag{5.6}$$

Keeping the notation of section 4, we have

Proposition 5.3.

$$\iota^*\Theta_\lambda = \Theta_{\iota^*\lambda} + \sum_{r=1}^{m-\kappa} \mathcal{C}_r^j \omega^r \wedge \omega_j + \textit{a constraint form}, \tag{5.7}$$

where

$$\mathcal{C}_r^j = \bar{L}_\alpha^j G_r^\alpha + \bar{L}_a^i \left(\Gamma_r^a \delta_i^j - \frac{\partial g_i^a}{\partial y_j^r} \right). \tag{5.8}$$

If Q is Lagrangian then $\iota^*\Theta_\lambda - \Theta_{\iota^*\lambda} \in \mathcal{I}$.
If Q is semi-holonomic then also $\iota^*\,\mathrm{d}\Theta_\lambda - \mathrm{d}\Theta_{\iota^*\lambda} \in \mathcal{I}$.

Remark 5.4. To justify correspondence with the formulae in paper [17], it is useful to compute the explicit form of some operators from [17] for our case of π-adapted constraints. In this way one obtains the following relations,

$$\mathcal{C}_{ri}^{\alpha j} = G_r^\alpha \delta_i^j, \qquad \mathcal{C}_{ri}^{aj} = \Gamma_r^a \delta_i^j - \frac{\partial g_i^a}{\partial y_j^r}, \tag{5.9}$$

hence (5.8) becomes

$$\mathcal{C}_r^j = \bar{L}_\alpha^i \mathcal{C}_{ri}^{\alpha j} + \bar{L}_a^i \mathcal{C}_{ri}^{aj} = \sum_{A=\{\alpha,a\}} \bar{L}_A^i \mathcal{C}_{ri}^{Aj}, \tag{5.10}$$

where summation runs over $\alpha = 1, \ldots, \kappa - k$ and $a = 1, \ldots, k$. Similarly, for the $\mathcal{C}$-modified Euler–Lagrange operator introduced in [17], we simply obtain $\mu_r = \varepsilon_r$.

Proof of proposition 5.3.

$$\begin{aligned}
\iota^*\Theta_\lambda &= \bar{L}\omega_0 + \sum_{r=1}^{m-\kappa} \left(\frac{\partial L}{\partial y_j^r} \circ \iota \right) \omega^r \wedge \omega_j + \bar{L}_\alpha^j \omega^{m-\kappa+\alpha} \wedge \omega_j + \bar{L}_a^j \bar{\omega}^{m-k+a} \wedge \omega_j \\
&= \bar{L}\omega_0 + \sum_{r=1}^{m-\kappa} \left(\left(\frac{\partial L}{\partial y_j^r} \circ \iota \right) + \bar{L}_\alpha^j G_r^\alpha + \bar{L}_a^j \Gamma_r^a \right) \omega^r \wedge \omega_j \\
&\quad + \left(\bar{L}_\alpha^j + \bar{L}_a^j \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^{m-\kappa+\alpha}} \right) \varphi^\alpha \wedge \omega_j + \bar{L}_a^j \varphi^a \wedge \omega_j.
\end{aligned} \tag{5.11}$$

From $\iota^*\,\mathrm{d}L = \mathrm{d}\bar{L}$ we obtain the relation

$$\frac{\partial \bar{L}}{\partial y_j^s} = \frac{\partial L}{\partial y_j^s} \circ \iota + \sum_{a=1}^{k} \bar{L}_a^i \frac{\partial g_i^a}{\partial y_j^s}, \qquad 1 \leqslant s \leqslant m-k. \tag{5.12}$$

Hence,

$$\begin{aligned}
\iota^*\Theta_\lambda &= \bar{L}\omega_0 + \sum_{r=1}^{m-\kappa} \left(\frac{\partial \bar{L}}{\partial y_j^r} - \bar{L}_a^i \frac{\partial g_i^a}{\partial y_j^r} + \bar{L}_a^j \Gamma_r^a + \bar{L}_\alpha^j G_r^\alpha \right) \omega^r \wedge \omega_j \\
&\quad + \left(\bar{L}_\alpha^j + \bar{L}_a^j \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^{m-\kappa+\alpha}} \right) \varphi^\alpha \wedge \omega_j + \bar{L}_a^j \varphi^a \wedge \omega_j \\
&= \Theta_{\iota^*\lambda} + \sum_{r=1}^{m-\kappa} \left(\bar{L}_a^i \left(\Gamma_r^a \delta_i^j - \frac{\partial g_i^a}{\partial y_j^r} \right) + \bar{L}_\alpha^j G_r^\alpha \right) \omega^r \wedge \omega_j + \text{a constraint form}.
\end{aligned} \tag{5.13}$$

If Q is Lagrangian, we get using theorem 4.12 that $\mathcal{C}_r^j = 0$, hence $\iota^*\Theta_\lambda - \Theta_{\iota^*\lambda} \in \mathcal{I}$. If Q is semi-holonomic then, moreover, $\mathrm{d}\varphi^a \in \mathcal{I}$, which means that $\iota^*\mathrm{d}\Theta_\lambda - \mathrm{d}\Theta_{\iota^*\lambda} = \mathrm{d}\bar{L}_a^j \wedge \varphi^a \wedge \omega_j + \bar{L}_a^j \mathrm{d}\varphi^a \wedge \omega_j \in \mathcal{I}$. □

Now, we get the following fibred chart expressions for a constrained system:

Theorem 5.5. *Every element of* $[[\iota^*\mathrm{d}\Theta_\lambda]]$ *takes the form*

$$\bar{\alpha} \approx \iota^*\mathrm{d}\Theta_\lambda \approx \sum_{s=1}^{m-k} \bar{A}_s\omega^s \wedge \omega_0 + \sum_{t,s=1}^{m-k} \bar{B}^{ij}_{ts}\omega^t \wedge \mathrm{d}y^s_j \wedge \omega_i$$
$$\approx \sum_{r=1}^{m-\kappa} \tilde{A}_r\omega^r \wedge \omega_0 + \sum_{r=1}^{m-\kappa}\sum_{s=1}^{m-k} \tilde{B}^{ij}_{rs}\omega^r \wedge \mathrm{d}y^s_j \wedge \omega_i, \tag{5.14}$$

where

$$\bar{A}_s = \left(A_s + A_{m-k+a}\frac{1}{n}\frac{\partial g^a_l}{\partial y^s_l} + \left(B^{il}_{s,m-k+b} + B^{il}_{m-k+a,m-k+b}\frac{1}{n}\frac{\partial g^a_p}{\partial y^s_p}\right)\frac{\mathrm{d}'_c g^b_l}{\mathrm{d}x^i}\right)\circ\iota, \quad 1 \leqslant s \leqslant m-k,$$
$$\bar{B}^{ij}_{ts} = \left(B^{ij}_{ts} + B^{ij}_{m-k+a,s}\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p} + B^{il}_{t,m-k+a}\frac{\partial g^a_l}{\partial y^s_j} + B^{il}_{m-k+a,m-k+b}\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p}\frac{\partial g^b_l}{\partial y^s_j}\right)\circ\iota,$$
$$1 \leqslant t, s \leqslant m-k, \quad 1 \leqslant i, j \leqslant n, \tag{5.15}$$

and

$$\tilde{A}_r = \bar{A}_r + \bar{A}_{m-\kappa+\alpha}G^\alpha_r$$
$$= \Big(A_r + A_{m-\kappa+\alpha}G^\alpha_r + A_{m-k+a}\Gamma^a_r + \big(B^{il}_{r,m-k+b} + B^{il}_{m-\kappa+\alpha,m-k+b}G^\alpha_r$$
$$+ B^{il}_{m-k+a,m-k+b}\Gamma^a_r\big)\frac{\mathrm{d}'_c g^b_l}{\mathrm{d}x^i}\Big)\circ\iota, \quad 1 \leqslant r \leqslant m-\kappa,$$
$$\tilde{B}^{ij}_{rs} = \bar{B}^{ij}_{rs} + \bar{B}^{ij}_{m-\kappa+\alpha,s}G^\alpha_r$$
$$= \Big(B^{ij}_{rs} + B^{ij}_{m-\kappa+\alpha,s}G^\alpha_r + B^{ij}_{m-k+a,s}\Gamma^a_r$$
$$+ \big(B^{il}_{r,m-k+b} + B^{il}_{m-\kappa+\alpha,m-k+b}G^\alpha_r + B^{il}_{m-k+a,m-k+b}\Gamma^a_r\big)\frac{\partial g^b_l}{\partial y^s_j}\Big)\circ\iota,$$
$$1 \leqslant r \leqslant m-\kappa, \quad 1 \leqslant s \leqslant m-k, \quad 1 \leqslant i, j \leqslant n. \tag{5.16}$$

Equivalently, in terms of a Lagrangian $\lambda = L\omega_0$,

$$\tilde{A}_r = \varepsilon'_r(\bar{L}) - \bar{L}^j_a\varepsilon'_r(g^a_j) - \mathcal{C}^{Ai}_{rj}\frac{\mathrm{d}'_c\bar{L}^j_A}{\mathrm{d}x^i}, \qquad 1 \leqslant r \leqslant m-\kappa,$$
$$\tilde{B}^{ij}_{rs} = -\frac{\partial^2\bar{L}}{\partial y^s_j\partial y^r_i} + \bar{L}^p_a\frac{\partial^2 g^a_p}{\partial y^s_j\partial y^r_i} - \mathcal{C}^{Ai}_{rp}\frac{\partial\bar{L}^p_A}{\partial y^s_j},$$
$$1 \leqslant r \leqslant m-\kappa, \quad 1 \leqslant s \leqslant m-k, \quad 1 \leqslant i, j \leqslant n. \tag{5.17}$$

Proof. First, let us prove (5.14), (5.15). With (5.2), (5.1) and (4.28) we have

$$\bar{\alpha} \approx \iota^*\,\mathrm{d}\Theta_\lambda$$
$$\approx \sum_{s=1}^{m-k}(A_s\circ\iota)\omega^s \wedge \omega_0 + \sum_{a=1}^{k}(A_{m-k+a}\circ\iota)\bar{\omega}^{m-k+a} \wedge \omega_0$$
$$+ \sum_{t,s=1}^{m-k}\big(B^{ij}_{ts}\circ\iota\big)\omega^t \wedge \mathrm{d}y^s_j \wedge \omega_i + \sum_{s=1}^{m-k}\big(B^{ij}_{m-k+a,s}\circ\iota\big)\bar{\omega}^{m-k+a} \wedge \mathrm{d}y^s_j \wedge \omega_i$$

$$+\sum_{s=1}^{m-k}\big(B^{ij}_{s,m-k+a}\circ\iota\big)\omega^s\wedge \mathrm{d}g^a_j\wedge\omega_i+\big(B^{ij}_{m-k+a,m-k+b}\circ\iota\big)\bar\omega^{m-k+a}\wedge \mathrm{d}g^b_j\wedge\omega_i$$

$$\approx\sum_{s=1}^{m-k}\left((A_s\circ\iota)+(A_{m-k+a}\circ\iota)\frac{1}{n}\frac{\partial g^a_l}{\partial y^s_l}\right)\omega^s\wedge\omega_0$$

$$+\sum_{t,s=1}^{m-k}\left(\big(B^{ij}_{ts}\circ\iota\big)+\big(B^{ij}_{m-k+a,s}\circ\iota\big)\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p}\right)\omega^t\wedge \mathrm{d}y^s_j\wedge\omega_i$$

$$+\sum_{t=1}^{m-k}\big(B^{il}_{t,m-k+a}\circ\iota\big)\omega^t\wedge\left(\frac{\mathrm{d}'_c g^a_l}{\mathrm{d}x^j}\mathrm{d}x^j+\sum_{s=1}^{m-k}\frac{\partial g^a_l}{\partial y^s_j}\mathrm{d}y^s_j\right)\wedge\omega_i$$

$$+\big(B^{il}_{m-k+a,m-k+b}\circ\iota\big)\sum_{t=1}^{m-k}\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p}\omega^t\wedge\left(\frac{\mathrm{d}'_c g^b_l}{\mathrm{d}x^j}\mathrm{d}x^j+\sum_{s=1}^{m-k}\frac{\partial g^b_l}{\partial y^s_j}\mathrm{d}y^s_j\right)\wedge\omega_i,$$

hence

$$\bar\alpha\approx\sum_{s=1}^{m-k}\Bigg((A_s\circ\iota)+(A_{m-k+a}\circ\iota)\frac{1}{n}\frac{\partial g^a_l}{\partial y^s_l}+\big(B^{il}_{s,m-k+a}\circ\iota\big)\frac{\mathrm{d}'_c g^a_l}{\mathrm{d}x^i}$$
$$+\big(B^{il}_{m-k+a,m-k+b}\circ\iota\big)\frac{1}{n}\frac{\partial g^a_p}{\partial y^s_p}\frac{\mathrm{d}'_c g^b_l}{\mathrm{d}x^i}\Bigg)\omega^s\wedge\omega_0$$
$$+\sum_{t,s=1}^{m-k}\Bigg(\big(B^{ij}_{ts}\circ\iota\big)+\big(B^{ij}_{m-k+a,s}\circ\iota\big)\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p}+\big(B^{il}_{t,m-k+a}\circ\iota\big)\frac{\partial g^a_l}{\partial y^s_j}$$
$$+\big(B^{il}_{m-k+a,m-k+b}\circ\iota\big)\frac{1}{n}\frac{\partial g^a_p}{\partial y^t_p}\frac{\partial g^b_l}{\partial y^s_j}\Bigg)\omega^t\wedge \mathrm{d}y^s_j\wedge\omega_i. \tag{5.18}$$

This gives us formulae (5.15). Formulae (5.16) now follow by expressing the $\omega^{m-\kappa+\alpha}$ by means of the constraint forms φ^α according to (4.28).

Next, using proposition 5.3, and the notation introduced so far, we obtain

$$\bar\alpha\approx\iota^*\mathrm{d}\Theta_\lambda=\mathrm{d}\iota^*\Theta_\lambda$$
$$\approx \mathrm{d}\Theta_{\iota^*\lambda}+\mathrm{d}\mathcal{C}^i_r\wedge\omega^r\wedge\omega_i-\mathcal{C}^i_r\mathrm{d}y^r_i\wedge\omega_0$$
$$+\left(\bar L^i_\alpha+\bar L^i_a\frac{1}{n}\frac{\partial g^a_l}{\partial y^{m-\kappa+\alpha}_l}\right)\mathrm{d}\varphi^\alpha\wedge\omega_i+\bar L^i_a\,\mathrm{d}\varphi^a\wedge\omega_i$$
$$\approx\left(\frac{\partial_c\bar L}{\partial y^r}-\frac{\mathrm{d}'_c}{\mathrm{d}x^j}\frac{\partial\bar L}{\partial y^r_j}-\frac{\mathrm{d}'_c\mathcal{C}^j_r}{\mathrm{d}x^j}\right)\omega^r\wedge\omega_0+\frac{\partial\bar L}{\partial y^{m-\kappa+\alpha}_i}\mathrm{d}y^{m-\kappa+\alpha}_i\wedge\omega_0$$
$$-\left(\frac{\partial^2\bar L}{\partial y^s_j\partial y^r_i}+\frac{\partial\mathcal{C}^i_r}{\partial y^s_j}\right)\omega^r\wedge\mathrm{d}y^s_j\wedge\omega_i-\mathcal{C}^i_r\mathrm{d}y^r_i\wedge\omega_0$$
$$+\left(\bar L^i_\alpha+\bar L^i_a\frac{1}{n}\frac{\partial g^a_l}{\partial y^{m-\kappa+\alpha}_l}\right)\Bigg(-\mathrm{d}y^{m-\kappa+\alpha}_i\wedge\omega_0+G^\alpha_r\mathrm{d}y^r_i\wedge\omega_0$$
$$+\frac{\mathrm{d}'_c G^\alpha_r}{\mathrm{d}x^i}\omega^r\wedge\omega_0+\frac{\partial G^\alpha_r}{\partial y^s_j}\omega^r\wedge\mathrm{d}y^s_j\wedge\omega_i\Bigg)$$
$$-\bar L^i_a\left(\frac{\partial_c g^a_i}{\partial y^r}-\frac{\mathrm{d}'_c\Gamma^a_r}{\mathrm{d}x^i}+\frac{1}{n}\frac{\partial g^a_l}{\partial y^{m-\kappa+\alpha}_l}\frac{\mathrm{d}'_c G^\alpha_r}{\mathrm{d}x^i}\right)\omega^r\wedge\omega_0$$

$$- \bar{L}_a^i \left(\frac{\partial g_i^a}{\partial y_j^s} - \frac{1}{n} \frac{\partial g_l^a}{\partial y_l^s} \delta_i^j \right) \mathrm{d}y_j^s \wedge \omega_0$$

$$- \bar{L}_a^j \left(\frac{\partial \Gamma_r^a}{\partial y_j^s} - \frac{1}{n} \frac{\partial g_l^a}{\partial y_l^{m-\kappa+\alpha}} \frac{\partial G_r^\alpha}{\partial y_j^s} \right) \mathrm{d}y_j^s \wedge \omega^r \wedge \omega_i$$

$$= \tilde{A}_r \omega^r \wedge \omega_0 + \tilde{B}_{rs}^{ij} \omega^r \wedge \mathrm{d}y_j^s \wedge \omega_i \tag{5.19}$$

(summation over $1 \leqslant r \leqslant m - \kappa$ and $1 \leqslant s \leqslant m - k$), with

$$\tilde{B}_{rs}^{ij} = -\frac{\partial^2 \bar{L}}{\partial y_i^r \partial y_j^s} + \bar{L}_a^p \frac{\partial^2 g_p^a}{\partial y_i^r \partial y_j^s} - G_r^\alpha \frac{\partial \bar{L}_\alpha^i}{\partial y_j^s} - \left(\Gamma_r^a \delta_p^i - \frac{\partial g_p^a}{\partial y_i^r} \right) \frac{\partial \bar{L}_a^p}{\partial y_j^s}$$

$$= -\frac{\partial^2 \bar{L}}{\partial y_i^r \partial y_j^s} + \bar{L}_a^p \frac{\partial^2 g_p^a}{\partial y_i^r \partial y_j^s} - \mathcal{C}_{rp}^{Ai} \frac{\partial \bar{L}_A^p}{\partial y_j^s}, \tag{5.20}$$

and

$$\tilde{A}_r = \frac{\partial_c \bar{L}}{\partial y^r} - \frac{\mathrm{d}_c'}{\mathrm{d}x^i} \frac{\partial \bar{L}}{\partial y_i^r} - \bar{L}_a^j \left(\frac{\partial_c g_j^a}{\partial y^r} - \frac{\mathrm{d}_c' \Gamma_r^a}{\mathrm{d}x^j} \right) + \bar{L}_\alpha^j \frac{\mathrm{d}_c' G_r^\alpha}{\mathrm{d}x^j} - \frac{\mathrm{d}_c' \mathcal{C}_r^j}{\mathrm{d}x^j}$$

$$= \frac{\partial_c \bar{L}}{\partial y^r} - \frac{\mathrm{d}_c'}{\mathrm{d}x^i} \frac{\partial \bar{L}}{\partial y_i^r} - \bar{L}_a^j \left(\frac{\partial_c g_j^a}{\partial y^r} - \frac{\mathrm{d}_c'}{\mathrm{d}x^i} \frac{\partial g_j^a}{\partial y_i^r} \right) + \bar{L}_a^i \frac{\mathrm{d}_c'}{\mathrm{d}x^j} \left(\Gamma_r^a \delta_i^j - \frac{\partial g_i^a}{\partial y_j^r} \right) + \bar{L}_\alpha^i \frac{\mathrm{d}_c' G_r^\alpha}{\mathrm{d}x^i} - \frac{\mathrm{d}_c' \mathcal{C}_r^j}{\mathrm{d}x^j}$$

$$= \varepsilon_r'(\bar{L}) - \bar{L}_a^j \varepsilon_r'\big(g_j^a\big) + \bar{L}_A^i \frac{\mathrm{d}_c' \mathcal{C}_{ri}^{Aj}}{\mathrm{d}x^j} - \frac{\mathrm{d}_c' \big(\bar{L}_A^i \mathcal{C}_{ri}^{Aj} \big)}{\mathrm{d}x^j}$$

$$= \varepsilon_r'(\bar{L}) - \bar{L}_a^j \varepsilon_r'\big(g_j^a\big) - \mathcal{C}_{rj}^{Ai} \frac{\mathrm{d}_c' \bar{L}_A^j}{\mathrm{d}x^i}, \tag{5.21}$$

as desired. □

Corollary 5.6.

(1) If Q is Lagrangian then

$$\alpha \approx \big(\varepsilon_r'(\bar{L}) - \bar{L}_a^j \varepsilon_r'\big(g_j^a\big) \big) \omega^r \wedge \omega_0 - \frac{\partial^2 \bar{L}}{\partial y_j^s \partial y_i^r} \omega^r \wedge \mathrm{d}y_j^s \wedge \omega_i. \tag{5.22}$$

(2) If Q is semi-holonomic then

$$\alpha \approx \varepsilon_r'(\bar{L}) \omega^r \wedge \omega_0 - \frac{\partial^2 \bar{L}}{\partial y_j^s \partial y_i^r} \omega^r \wedge \mathrm{d}y_j^s \wedge \omega_i. \tag{5.23}$$

Proof. From theorem 4.12 we get that for a Lagrangian constraint $\mathcal{C}_{rj}^{Ai} = 0$ and $\partial^2 g_l^a \big/ \partial y_i^r \partial y_j^s = 0$. If Q is semi-holonomic then by theorem 4.15 also $\varepsilon_r'\big(g_j^a\big) = 0$. □

Definition 5.7. *Let λ be a Lagrangian, Q a π-adapted constraint on J^1Y, and $[[\iota^* \mathrm{d}\Theta_\lambda]]$ the corresponding constrained system. A (local) section $\gamma : X \to Y$ is called a* constrained extremal *of λ if $J^1\gamma$ is an integral section of the canonical distribution $\mathcal{C}$, and*

$$J^1\gamma^* i_\xi \iota^* \mathrm{d}\Theta_\lambda = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}. \tag{5.24}$$

Equations (5.24) are called constrained Euler–Lagrange equations.

Note that instead of (5.24) we can equivalently write

$$J^1\gamma^* i_\xi \bar{\alpha} = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \tag{5.25}$$

where $\bar{\alpha}$ is *any* element of $[[\iota^* \mathrm{d}\Theta_\lambda]]$.

By theorem 5.5, the constrained Euler–Lagrange equations in fibred coordinates take the form

$$\left(\varepsilon_r(\bar{L}) - \bar{L}^j_a\varepsilon_r\big(g^a_j\big) - \mathcal{C}^{Ai}_{rj}\frac{\mathrm{d}_c\bar{L}^j_A}{\mathrm{d}x^i}\right)\circ J^2\gamma = 0, \quad 1 \leqslant r \leqslant m-\kappa. \tag{5.26}$$

For Lagrangian constraints we have

$$\big(\varepsilon_r(\bar{L}) - \bar{L}^j_a\varepsilon_r\big(g^a_j\big)\big)\circ J^2\gamma = 0, \quad 1 \leqslant r \leqslant m-\kappa. \tag{5.27}$$

For semi-holonomic constraints we have

$$\varepsilon_r(\bar{L})\circ J^2\gamma = 0, \qquad 1 \leqslant r \leqslant m-\kappa. \tag{5.28}$$

Remark 5.8. We denote

$$\mathcal{E}_r\big(\bar{L}, \bar{L}^j_a\big) = \tilde{A}_r + \sum_{s=1}^{m-k}\tilde{B}^{ij}_{rs}y^s_{ij} = \varepsilon_r(\bar{L}) - \bar{L}^j_a\varepsilon_r\big(g^a_j\big) - \mathcal{C}^{Ai}_{rj}\frac{\mathrm{d}_c\bar{L}^j_A}{\mathrm{d}x^i}, \quad 1 \leqslant r \leqslant m-\kappa, \tag{5.29}$$

and call this operator the *constraint Euler–Lagrange operator*. We can see that for general (non-integrable) constrained systems functions (5.29) generalizing the Euler–Lagrange expressions depend upon the 'constrained Lagrangian' $\bar{L} = L\circ\iota$ and other κn functions $\bar{L}^j_A$ (which cannot be obtained by means of $\bar{L}$). In this way, we can expect that a 'constrained variational principle' will (similarly as in mechanics) involve not merely a single function but rather $1+\kappa n$ functions (more precisely, a differential form with $1+\kappa n$ components) (cf [15, 28]

6. Constrained Hamilton–De Donder equations, regularity of constrained systems

Let λ be a Lagrangian, Q a π-adapted constraint on J^1Y. Consider the constrained system $[[\iota^*\,\mathrm{d}\Theta_\lambda]]$.

Definition 6.1. *For* $\bar{\alpha}\in[[\iota^*\mathrm{d}\Theta_\lambda]]$ *we consider the ideal* $\mathcal{H}_{\bar{\alpha}}$ *in the exterior algebra on* Q*, generated by* n*-forms*

$$i_\xi\bar{\alpha}, \qquad \text{where} \quad \xi \text{ runs over all } \pi_1\text{-vertical vector fields on } Q \text{ belonging to } \mathcal{C}. \tag{6.1}$$

$\mathcal{H}_{\bar{\alpha}}$ *will be called the* constrained Hamiltonian ideal *of* $\bar{\alpha}$*. (Local) sections* $\delta : X\to Q$ *which are integral sections of* $\mathcal{H}_{\bar{\alpha}}$ *and the constraint ideal* $\mathcal{I}$ *will be called* constrained Hamilton extremals *of the* $(n+1)$*-form* $\bar{\alpha}$*. Equations for constrained Hamilton extremals of* $\bar{\alpha}$*, i.e.,*

$$\delta^*\varphi^A = 0, \qquad \delta^*i_\xi\bar{\alpha} = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi\in\mathcal{C}, \tag{6.2}$$

will be called constrained Hamilton equations.

Note that

(1) Constrained Hamilton equations (6.2) *do not depend on the choice of a constraint form* φ in (5.3).
(2) Constrained Euler–Lagrange equations (5.24) are equations for *holonomic* integral sections of *any* Hamiltonian ideal $\mathcal{H}_{\bar{\alpha}}$, where $\bar{\alpha}\in[[\iota^*\,\mathrm{d}\Theta_\lambda]]$.
(3) Considering different *classes*

$$\bar{\alpha} \bmod \mathcal{I} \tag{6.3}$$

provides different constrained Hamilton equations. For the elements of $\bar{\alpha} \bmod \mathcal{I}$ the constrained Hamilton equations are the same.

Denote by $[[\iota^* \mathrm{d}\Theta_\lambda]]_Y$ the class of forms

$$\bar{\alpha}' = \iota^* \mathrm{d}\Theta_\lambda + \bar{F} + \varphi, \quad \text{where } \bar{F} \text{ is 2-contact and } \pi_{1,0}\text{-horizontal, and } \varphi \in \mathcal{I}. \tag{6.4}$$

Definition 6.2. *The class* $[[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ *will be called the* constrained Hamilton–De Donder system *of* λ. *Constrained Hamilton equations of* $\bar{\alpha} \in [[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ *will be called* constrained Hamilton–De Donder equations.

Similarly as in section 3 we can introduce the concept of regularity for constrained Hamilton–De Donder systems:

Definition 6.3. *An* $(n+1)$*-form* $\bar{\alpha} \in [[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ *is called* regular *if a system of generators of* $\mathcal{H}_{\bar{\alpha}}$ *has the maximal rank (i.e., equal to* $m - \kappa + (m-k)n$*). A Lagrangian constrained system on* Q *is called* De Donder regular *if in the class* $[[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ *there exists a regular representative.*

Theorem 6.4. *The constrained system* $[[\iota^*\mathrm{d}\Theta_\lambda]]$ *is De Donder regular if and only if one of the following equivalent conditions holds,*

$$\operatorname{rank}\big(\tilde{B}^{ij}_{rs}\big) = \max = (m-\kappa)n, \tag{6.5}$$

$$\operatorname{rank}\left(\frac{\partial^2 \bar{L}}{\partial y^r_i \partial y^s_j} - \bar{L}^p_a \frac{\partial^2 g^a_p}{\partial y^r_i \partial y^s_j} + C^{Ai}_{rp}\frac{\partial \bar{L}^p_A}{\partial y^s_j}\right) = (m-\kappa)n, \tag{6.6}$$

$$\operatorname{rank}\left(\bar{B}^{ij}_{rs} + \bar{B}^{ij}_{m-\kappa+\alpha,s}G^\alpha_r\right) = (m-\kappa)n, \tag{6.7}$$

with $\bar{B}^{ij}_{ts}$, $1 \leqslant t, s \leqslant k$, *defined by (5.15).*

If $[[\iota^*\mathrm{d}\Theta_\lambda]]$ *is De Donder regular then every form* $\bar{\alpha} \in [[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ *is regular. Consequently,*

(1) every constrained Hamilton extremal of $\bar{\alpha}$ *is holonomic,*
(2) constrained Hamilton equations of $\bar{\alpha}$ *are equivalent to the constrained Euler–Lagrange equations,*
(3) constrained Hamilton equations of all $\bar{\alpha}$ *(though different) are equivalent, i.e., have the same solutions,*
(4) every constrained Hamilton extremal of $\bar{\alpha}$ *is a prolongation of a constrained extremal.*

Proof. First note that for every $\pi_{1,0}$-horizontal 2-contact form F on Q one has

$$F = F^i_{\sigma\nu}\bar{\omega}^\sigma \wedge \bar{\omega}^\nu \wedge \omega_i = \sum_{q,r=1}^{m-\kappa} \tilde{F}^i_{qr}\omega^q \wedge \omega^r \wedge \omega_i + \text{a constraint form}. \tag{6.8}$$

If $\bar{\alpha} \in [[\iota^* \mathrm{d}\Theta_\lambda]]_Y$, we have

$$\bar{\alpha} = \tilde{A}_r\omega^r \wedge \omega_0 + \tilde{F}^i_{qr}\omega^q \wedge \omega^r \wedge \omega_i + \tilde{B}^{ij}_{rs}\omega^r \wedge \mathrm{d}y^s_j \wedge \omega_i + \varphi, \tag{6.9}$$

where $\tilde{A}_r$ and $\tilde{B}^{ij}_{rs}$ are given by (5.16) or (5.17), $\tilde{F}^i_{qr} = -\tilde{F}^i_{rq}$, and $\varphi \in \mathcal{I}$. Computing generators (6.1) of $\mathcal{H}_{\bar{\alpha}}$ we obtain the following system of $m - \kappa + (m-k)n$ differential n-forms:

$$\tilde{A}_r\omega_0 + 2\tilde{F}^i_{rq}\omega^q \wedge \omega_i + \tilde{B}^{ij}_{rs}\mathrm{d}y^s_j \wedge \omega_i, \qquad \tilde{B}^{ij}_{rs}\omega^r \wedge \omega_i. \tag{6.10}$$

Hence, the matrix of generators of $\mathcal{H}_{\bar{\alpha}}$ is the following matrix with $m - \kappa + (m-k)n$ rows and $1 + (m-\kappa)n + (m-k)n^2$ columns:

$$\begin{pmatrix} \tilde{A}_r & 2\tilde{F}^i_{rq} & \tilde{B}^{ij}_{rs} \\ 0 & \tilde{B}^{ij}_{rs} & 0 \end{pmatrix}. \tag{6.11}$$

If $\mathcal{H}_{\bar\alpha}$ is regular, i.e., the above matrix has the maximal rank, then the matrix $\big(\tilde B^{ij}_{rs}\big)$ has the maximal rank, i.e. equal to $(m-\kappa)n$. Conversely, if $\operatorname{rank}\big(\tilde B^{ij}_{rs}\big) = \max = (m-\kappa)n$ then the rank of (6.11) is maximal. Indeed, since all columns of $\big(\tilde B^{ij}_{rs}\big)$ (labelled by (r,i)) are linearly independent, for every fixed i the matrix $\big(\tilde B^{ij}_{rs}\big)$ with $m-\kappa$ columns labelled by r, and $(m-k)n$ rows labelled by (s,j), has the maximal rank, $m-\kappa$. Consequently, the matrix $\big(\tilde B^{ij}_{rs}\big)$ with $m-\kappa$ rows labelled by r and $(m-k)n^2$ columns labelled by (i,s,j), appearing in the right upper corner of (6.11), has rank $m-\kappa$. This proves that the corresponding form $\bar\alpha$ is regular. Moreover, we can see that regularity does not depend on the choice of functions $\bar F^i_{qr}$, i.e., of $\bar\alpha$ mod $\mathcal{I}$ in the class $[[\iota^*\,\mathrm{d}\Theta_\lambda]]_Y$.

Let us prove (1)–(4). Assume that δ is a local section of $Q \to X$ annihilating all the forms (6.10). If $\operatorname{rank}\tilde B^{ij}_{rs} = (m-\kappa)n$ then $\delta^*\big(\tilde B^{ij}_{rs}\omega^r \wedge \omega_i\big) = 0$ means that $\delta^*(\omega^r \wedge \omega_i) = 0$ for all r, i, i.e.,

$$0 = \big(\mathrm{d}(y^r\circ\delta) - \big(y^r_j\circ\delta\big)\,\mathrm{d}x^j\big)\wedge\omega_i = \left(\frac{\partial(y^r\circ\delta)}{\partial x^j}\mathrm{d}x^j - \big(y^r_j\circ\delta\big)\mathrm{d}x^j\right)\wedge\omega_i$$

$$= \left(\frac{\partial(y^r\circ\delta)}{\partial x^i} - \big(y^r_i\circ\delta\big)\right)\omega_0 \quad\Leftrightarrow\quad y^r_i\circ\delta = \frac{\partial(y^r\circ\delta)}{\partial x^i},\quad 1\leqslant r\leqslant m-\kappa. \tag{6.12}$$

The condition that δ is also an integral section of $\mathcal{C}$ then means that $\delta^*\varphi^A = 0$ for all A, i.e.,

$$\delta^*\omega^{m-\kappa+\alpha} = 0, \qquad \delta^*\omega^{m-k+a} = 0, \tag{6.13}$$

proving that every solution of $\mathcal{H}_{\bar\alpha}$, which is an integral section of $\mathcal{C}$, is holonomic (and, indeed, satisfies the equations of constraints).

If δ is a solution of constrained Hamilton equations of $\bar\alpha$ then by (1), $\delta = J^1\gamma$ for a section γ of π. Hence, for every π_1-vertical vector field $\xi \in \mathcal{C}$, $0 = \delta^* i_\xi\bar\alpha = J^1\gamma^* i_\xi\bar\alpha = J^1\gamma^* i_\xi\iota^*\,\mathrm{d}\Theta_\lambda$, i.e., γ is a constrained extremal, and we get a bijective correspondence between solutions of the constrained Euler–Lagrange equations and any associated constrained Hamilton–De Donder equations. □

Corollary 6.5. *If Q is a Lagrangian constraint, or, if Q is a semi-holonomic constraint then the regularity conditions (6.5)–(6.7) read*

$$\det\big(\bar B^{ij}_{rs}\big) = -\det\left(\frac{\partial^2\bar L}{\partial y^s_j\partial y^r_i}\right) \neq 0. \tag{6.14}$$

7. Non-holonomic Legendre transformation

Theorem 7.1. *Consider a Lagrangian λ and a π-adapted constraint $Q \subset J^1Y$. Let $[[\iota^* d\Theta_\lambda]]_Y$ be the related constrained Hamilton–De Donder system. Let $x \in Q$ be a point. Suppose that in a neighbourhood of x,*

$$\frac{\partial\tilde B^{ij}_{rs}}{\partial y^t_l} = \frac{\partial\tilde B^{il}_{rt}}{\partial y^s_j}, \qquad 1\leqslant r\leqslant m-\kappa, \qquad 1\leqslant s,t\leqslant m-k. \tag{7.1}$$

Then there exists a neighbourhood $U \subset Q$ of x, and, on U, functions P^i_r, and a n-form η, such that the class $[[\iota^*\mathrm{d}\Theta_\lambda]]_Y$ has a representative of the form

$$\bar\alpha = \eta\wedge\omega_0 + \mathrm{d}P^i_r\wedge\mathrm{d}y^r\wedge\omega_i. \tag{7.2}$$

If, moreover, the constrained system $[[\iota^*\mathrm{d}\Theta_\lambda]]$ is De Donder regular then the map $\big(x^i, y^\sigma, y^r_i, y^{m-\kappa+\alpha}_i\big) \to \big(x^i, y^\sigma, P^i_r, y^{m-\kappa+\alpha}_i\big)$ is a coordinate transformation on U.

Proof. Condition (7.1) guarantees that in a neighbourhood $U \subset Q$ of x there are functions P^i_r such that

$$\tilde{B}^{ij}_{rs} = -\frac{\partial P^i_r}{\partial y^s_j}. \tag{7.3}$$

Hence for elements of the class $[[\iota^* \, \mathrm{d}\Theta_\lambda]]_Y$ we obtain using theorem 5.5 and (6.4)

$$\begin{aligned}
\bar{\alpha} &\approx \iota^* \, \mathrm{d}\Theta_\lambda \approx \tilde{A}_r \omega^r \wedge \omega_0 + \frac{\partial P^i_r}{\partial y^s_j} \mathrm{d}y^s_j \wedge \omega^r \wedge \omega_i \\
&\approx \tilde{A}_r \omega^r \wedge \omega_0 + \mathrm{d}P^i_r \wedge \omega^r \wedge \omega_i - \frac{\mathrm{d}'_c P^i_r}{\mathrm{d}x^j} \mathrm{d}x^j \wedge \omega^r \wedge \omega_i \\
&= \left(\tilde{A}_r + \frac{\mathrm{d}'_c P^i_r}{\mathrm{d}x^i} \right) \mathrm{d}y^r \wedge \omega_0 - y^r_i \, \mathrm{d}P^i_r \wedge \omega_0 + \mathrm{d}P^i_r \wedge \mathrm{d}y^r \wedge \omega_i \\
&\approx \left(\tilde{A}_r + \frac{\mathrm{d}'_c P^i_r}{\mathrm{d}x^i} - y^q_i \frac{\partial_c P^i_q}{\partial y^r} \right) \mathrm{d}y^r \wedge \omega_0 - y^r_i \frac{\partial P^i_r}{\partial y^s_j} \mathrm{d}y^s_j \wedge \omega_0 + \mathrm{d}P^i_r \wedge \mathrm{d}y^r \wedge \omega_i .
\end{aligned}$$

In this way, we have obtained a representative

$$\bar{\alpha} = \left(\tilde{A}_r + \frac{\mathrm{d}'_c P^i_r}{\mathrm{d}x^i} - y^q_i \frac{\partial_c P^i_q}{\partial y^r} \right) \mathrm{d}y^r \wedge \omega_0 - y^r_i \frac{\partial P^i_r}{\partial y^s_j} \mathrm{d}y^s_j \wedge \omega_0 + \mathrm{d}P^i_r \wedge \mathrm{d}y^r \wedge \omega_i . \tag{7.4}$$

Denote

$$\bar{\alpha} = \eta \wedge \omega_0 + \mathrm{d}P^i_r \wedge \mathrm{d}y^r \wedge \omega_i , \tag{7.5}$$

with

$$\eta = \bar{\eta}_j \mathrm{d}x^j + \bar{\eta}_r \mathrm{d}y^r + \bar{\eta}^j_s \mathrm{d}y^s_j , \tag{7.6}$$

where $\bar{\eta}_j$, $1 \leqslant j \leqslant n$, are arbitrary functions on U, and

$$\begin{aligned}
\bar{\eta}_r &= \tilde{A}_r + \frac{\mathrm{d}'_c P^i_r}{\mathrm{d}x^i} - \sum_{q=1}^{m-\kappa} y^q_i \frac{\partial_c P^i_q}{\partial y^r}, \qquad 1 \leqslant r \leqslant m-\kappa, \\
\bar{\eta}^j_s &= -y^r_i \frac{\partial P^i_r}{\partial y^s_j}, \qquad 1 \leqslant s \leqslant m-k, \quad 1 \leqslant j \leqslant n.
\end{aligned} \tag{7.7}$$

Finally, by (7.3), the regularity condition (6.5) (which means that $\bar{\alpha}$ is De Donder regular) coincides with the regularity condition for the map $\left(x^i, y^\sigma, y^r_i, y^{m-\kappa+\alpha}_i\right) \to \left(x^i, y^\sigma, P^i_r, y^{m-\kappa+\alpha}_i\right)$. □

Remark 7.2. With the help of (5.17) one can rewrite the integrability condition (7.1) in terms of a Lagrangian and the constraint functions as follows:

$$\frac{\partial \bar{L}^p_a}{\partial y^t_l} \frac{\partial^2 g^a_p}{\partial y^s_j \partial y^r_i} - \frac{\partial \mathcal{C}^{Ai}_{rp}}{\partial y^t_l} \frac{\partial \bar{L}^p_A}{\partial y^s_j} = \frac{\partial \bar{L}^p_a}{\partial y^s_j} \frac{\partial^2 g^a_p}{\partial y^t_l \partial y^r_i} - \frac{\partial \mathcal{C}^{Ai}_{rp}}{\partial y^s_j} \frac{\partial \bar{L}^p_A}{\partial y^t_l}. \tag{7.8}$$

Let us find explicit formulae for the functions P^i_r in (7.3).

Proposition 7.3. *Let $x \in U$, and consider a mapping $\chi : [0,1] \times W \to W$ defined by*

$$\left(u, x^i, y^\sigma, y^s_j\right) \to \left(x^i, y^\sigma, u y^s_j\right), \tag{7.9}$$

where $W \subset U \subset Q$ is an appropriate neighbourhood of x. Then for arbitrary functions $\psi^i_r(x^j, y^\nu)$ (respectively $\tilde{\psi}^i_r(x^j, y^\nu)$), $1 \leqslant r \leqslant m-\kappa$, $1 \leqslant i \leqslant n$, the functions

$$\begin{aligned} P^i_r &= -y^s_j \int_0^1 \left(\tilde{B}^{ij}_{rs} \circ \chi\right) \mathrm{d}u + \psi^i_r(x^j, y^\nu) \\ &= \frac{\partial \bar{L}}{\partial y^r_i} + y^s_j \int_0^1 \left(C^{Ai}_{rp} \frac{\partial \bar{L}^p_A}{\partial y^s_j} - \bar{L}^p_a \frac{\partial^2 g^a_p}{\partial y^s_j \partial y^r_i} \right) \circ \chi \,\mathrm{d}u + \tilde{\psi}^i_r, \quad 1 \leqslant r \leqslant m-\kappa, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{7.10}$$

are solutions of (7.3).

Proof. Integrability condition (7.1) for the $\bar{B}^{ij}_{rs}$ ensures that in a neighbourhood of every point in U one can find solutions of (7.3) by the Poincaré lemma. Put

$$P^i_r = -y^s_j \int_0^1 \left(\tilde{B}^{ij}_{rs} \circ \chi\right) \mathrm{d}u + \psi^i_r, \tag{7.11}$$

where the ψ^i_r do not depend on the y^s_j. Then, indeed, with the help of (7.1),

$$\frac{\partial P^i_r}{\partial y^s_j} = -\int_0^1 \left(\tilde{B}^{ij}_{rs} \circ \chi\right) \mathrm{d}u - y^t_l \int_0^1 \left(\frac{\partial \tilde{B}^{il}_{rt}}{\partial y^s_j} \circ \chi \right) u \,\mathrm{d}u = -\int_0^1 \mathrm{d}\left(u\left(\tilde{B}^{ij}_{rs} \circ \chi\right)\right) = -\tilde{B}^{ij}_{rs},$$

as desired.

Using formula (5.17), equation (7.11) takes the form

$$\begin{aligned} P^i_r &= y^s_j \int_0^1 \left(\frac{\partial^2 \bar{L}}{\partial y^s_j \partial y^r_i} + C^{Ai}_{rp} \frac{\partial \bar{L}^p_A}{\partial y^s_j} - \bar{L}^p_a \frac{\partial^2 g^a_p}{\partial y^s_j \partial y^r_i} \right) \circ \chi \,\mathrm{d}u + \psi^i_r \\ &= \frac{\partial \bar{L}}{\partial y^r_i} + y^s_j \int_0^1 \left(C^{Ai}_{rp} \frac{\partial \bar{L}^p_A}{\partial y^s_j} - \bar{L}^p_a \frac{\partial^2 g^a_p}{\partial y^s_j \partial y^r_i} \right) \circ \chi \,\mathrm{d}u + \tilde{\psi}^i_r, \end{aligned} \tag{7.12}$$

since

$$\begin{aligned} \int_0^1 \mathrm{d}\left(\frac{\partial \bar{L}}{\partial y^r_i} \circ \chi \right) &= \left[\frac{\partial \bar{L}}{\partial y^r_i} \circ \chi \right]_{u=0}^{u=1} = \frac{\partial \bar{L}}{\partial y^r_i} - f^i_r(x^j, y^\nu) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{\partial \bar{L}}{\partial y^r_i} \circ \chi \right) \mathrm{d}u = y^s_j \int_0^1 \left(\frac{\partial^2 \bar{L}}{\partial y^s_j \partial y^r_i} \circ \chi \right) \mathrm{d}u. \end{aligned}$$

This completes the proof. □

Definition 7.4. *The form $\bar{\alpha}$ (7.2) will be called the* canonical representative *of the constrained Hamilton–De Donder system $[[\iota^* \mathrm{d}\Theta_\lambda]]_Y$.*

Functions P^i_r (7.10) will be called constraint momenta*, and the local coordinate transformation $\left(x^i, y^\sigma, y^r_i, y^{m-\kappa+\alpha}_i\right) \to \left(x^i, y^\sigma, P^i_r, y^{m-\kappa+\alpha}_i\right)$ on Q the* constraint Legendre transformation*. (Any) 1-form η in (7.2) (given by (7.6), (7.7)) will be called* energy 1-form.

For the constrained Hamilton–De Donder system $[[\iota^* \mathrm{d}\Theta_\lambda]]_Y$ we have *a family of energy 1-forms* $\eta + \varphi$ where η are given by (7.6), (7.7) and φ runs over constraint 1-forms in $\mathcal{I}$. In general, energy 1-form *need not be closed.*

To compute *constrained Hamilton–De Donder equations* in constraint Legendre coordinates we have to express in these coordinates the canonical representative $\bar{\alpha}$. From (7.5) it is clear that it is sufficient to transform η. Let us denote by

$$\eta = \eta_j \mathrm{d}x^j + \eta_r \mathrm{d}y^r + \eta^q_i \mathrm{d}P^i_q + \eta^j_{m-\kappa+\alpha} \mathrm{d}y^{m-\kappa+\alpha}_j \tag{7.13}$$

the chart expression of η in constraint Legendre coordinates. We have

$$\eta \approx \eta_j \mathrm{d}x^j + \eta_r \mathrm{d}y^r + \eta_i^q \left(\frac{\mathrm{d}'_c P_q^i}{\mathrm{d}x^j} \mathrm{d}x^j + \frac{\partial_c P_q^i}{\partial y^r} \omega^r + \frac{\partial P_q^i}{\partial y_j^s} \mathrm{d}y_j^s \right) + \eta^j_{m-\kappa+\alpha} \mathrm{d}y_j^{m-\kappa+\alpha}$$

$$= \left(\eta_j + \eta_i^q \frac{\mathrm{d}'_c P_q^i}{\mathrm{d}x^j} - \eta_i^q \frac{\partial_c P_q^i}{\partial y^r} y_j^r \right) \mathrm{d}x^j + \left(\eta_r + \eta_i^q \frac{\partial_c P_q^i}{\partial y^r} \right) \mathrm{d}y^r$$

$$+ \eta_i^q \frac{\partial P_q^i}{\partial y_j^r} \mathrm{d}y_j^r + \left(\eta_i^q \frac{\partial P_q^i}{\partial y_j^{m-\kappa+\alpha}} + \eta^j_{m-\kappa+\alpha} \right) \mathrm{d}y_j^{m-\kappa+\alpha}.$$

Comparing with (7.6) and (7.7) we can see that

$$\begin{aligned}
\bar{\eta}_r &= \eta_r + \eta_i^q \frac{\partial_c P_q^i}{\partial y^r} = \tilde{A}_r + \frac{\mathrm{d}'_c P_r^i}{\mathrm{d}x^i} - y_i^q \frac{\partial_c P_q^i}{\partial y^r} \\
\bar{\eta}_r^j &= \eta_l^q \frac{\partial P_q^l}{\partial y_j^r} = -y_i^q \frac{\partial P_q^i}{\partial y_j^r} \\
\bar{\eta}^j_{m-\kappa+\alpha} &= \eta_i^q \frac{\partial P_q^i}{\partial y_j^{m-\kappa+\alpha}} + \eta^j_{m-\kappa+\alpha} = -y_i^q \frac{\partial P_q^i}{\partial y_j^{m-\kappa+\alpha}}.
\end{aligned} \tag{7.14}$$

Now,

$$\left(\eta_l^q + y_i^q\right) \frac{\partial P_q^i}{\partial y_j^r} = 0, \qquad \text{i.e.,} \quad \eta_i^q = -y_i^q\left(x^j, y^\nu, P_r^j, y_j^{m-\kappa+\alpha}\right), \tag{7.15}$$

since the matrix $\left(\partial P_q^i / \partial y_j^r\right)$ is regular. Using the above relation we obtain

$$\eta_r = \bar{\eta}_r + y_i^q \frac{\partial_c P_q^i}{\partial y^r} = \tilde{A}_r + \frac{\mathrm{d}'_c P_r^i}{\mathrm{d}x^i}, \qquad \eta^j_{m-\kappa+\alpha} = 0, \tag{7.16}$$

(considered as functions in constraint Legendre coordinates).

Theorem 7.5. *Constrained Hamilton–De Donder equation (6.2) in constraint Legendre coordinates takes, for every canonical representative $\bar{\alpha} + \varphi$ where $\varphi \in \mathcal{I}$, the form*

$$\frac{\partial\left(P_r^i \circ \delta\right)}{\partial x^i} = \eta_r \circ \delta, \qquad \frac{\partial(y^r \circ \delta)}{\partial x^i} = -\eta_i^r \circ \delta, \qquad 1 \leqslant r \leqslant m - \kappa, \qquad 1 \leqslant i \leqslant n, \tag{7.17}$$

together with (6.13).

Proof. Taking into account (7.16), it is sufficient to compute the condition $\delta^* i_\xi \bar{\alpha} = 0$ for $\bar{\alpha}$ (7.2) with

$$\eta = \eta_j \,\mathrm{d}x^j + \eta_r \,\mathrm{d}y^r + \eta_i^r \,\mathrm{d}P_r^i, \tag{7.18}$$

and the vector fields $\partial_c/\partial y^r$ and $\partial/\partial P_r^i$ belonging to $\mathcal{C}$. This, however, leads to equations (7.17). □

Remark 7.6. If $\bar{\alpha}' \in [[\iota^* \,\mathrm{d}\Theta_\lambda]]_Y$, $\bar{\alpha}' = \bar{\alpha} + \bar{F} + \varphi$ is any other representative, the corresponding constrained Hamilton–De Donder equations take the canonical form

$$\begin{aligned}
\frac{\partial\left(P_r^i \circ \delta\right)}{\partial x^i} &= \eta_r \circ \delta + 2\left(F_{rq}^i \circ \delta\right) \left(\frac{\partial(y^q \circ \delta)}{\partial x^i} - \left(y_i^q \circ \delta\right) \right), \\
\frac{\partial(y^r \circ \delta)}{\partial x^i} &= -\eta_i^r \circ \delta.
\end{aligned} \tag{7.19}$$

Due to (7.15) equations (7.19) are equivalent to (7.17), as expected.

Remark 7.7. It is interesting that the canonical representative of the constrained Hamilton–De Donder equations is not a constrained Poincaré–Cartan $(n+1)$-form (as probably one might expect), but rather the form $\bar{\alpha}$ (7.2).

Taking into account results on Lagrangian and semi-holonomic constraints, we easily conclude the following:

Proposition 7.8. *For Lagrangian constraints and semi-holonomic constraints the integrability condition (7.1) is satisfied identically. Constraint momenta are given simply by formula*

$$P_r^i = \frac{\partial \bar{L}}{\partial y_i^r}, \qquad 1 \leqslant r \leqslant k, \qquad 1 \leqslant i \leqslant n. \tag{7.20}$$

The regularity condition takes the form

$$\det\left(\frac{\partial^2 \bar{L}}{\partial y_j^s \partial y_i^r}\right) \neq 0, \tag{7.21}$$

and the constraint Legendre transformation is a local map $(x^i, y^\sigma, y_i^r) \to (x^i, y^\sigma, P_r^i)$ *on the constraint* Q. *Moreover, if the constraint is semi-holonomic then the family of energy 1-forms* $\eta \bmod \mathcal{I}$ *contains a closed 1-form equal to* $-\mathrm{d}\bar{H}$, *where*

$$\bar{H} = -\bar{L} + P_r^i y_i^r. \tag{7.22}$$

Proof. The only non-trivial part of the proof is to show that for a semi-holonomic constraint $-\mathrm{d}\bar{H} - \eta \in \mathcal{I}$. If Q is semi-holonomic then $\iota^* \,\mathrm{d}\Theta_\lambda = \mathrm{d}\Theta_{\iota^*\lambda}$ up to a constraint $(n+1)$-form, and we get for every representative $\bar{\alpha} \in [[\iota^* \,\mathrm{d}\Theta_\lambda]]_Y$,

$$\bar{\alpha} \approx \iota^* \,\mathrm{d}\Theta_\lambda \approx \mathrm{d}\Theta_{\iota^*\lambda} = \mathrm{d}\left(\bar{L}\omega_0 + \frac{\partial \bar{L}}{\partial y_j^s}\omega^s \wedge \omega_j\right) = -\mathrm{d}\bar{H} \wedge \omega_0 + \mathrm{d}P_s^j \wedge \mathrm{d}y^s \wedge \omega_j. \tag{7.23}$$

Hence $-\mathrm{d}\bar{H} \wedge \omega_0 \approx \eta \wedge \omega_0$, meaning that among energy 1-forms one has $\eta = -\mathrm{d}\bar{H}$. □

8. Illustrative examples

Example 8.1. On the fibred manifold $\pi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with canonical coordinates (x^1, x^2, y^1, y^2), consider a Lagrange function

$$L = y_1^1 y_2^2 + y_2^1 y_1^2. \tag{8.1}$$

L gives rise to a first-order Lagrangian system represented by the 3-form $\alpha \sim \mathrm{d}\Theta_\lambda$,

$$\alpha = A_\sigma \omega^\sigma \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + B_{\sigma\nu}^{ji} \omega^\sigma \wedge \mathrm{d}y_i^\nu \wedge \omega_j, \tag{8.2}$$

where by (2.13), $A_\sigma = 0$, $\sigma = 1, 2$, and

$$B_{\sigma\nu}^{ji} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{8.3}$$

Euler–Lagrange equations take the form

$$\frac{\partial^2 y^1}{\partial x^1 \partial x^2} = 0, \qquad \frac{\partial^2 y^2}{\partial x^1 \partial x^2} = 0. \tag{8.4}$$

Lagrangian (8.1) is De Donder regular, since $\det\left(B^{ij}_{\sigma\nu}\right) \neq 0$ (cf theorem 3.4). Legendre transformation is a diffeomorphism

$$\left(x^1, x^2, y^1, y^2, y^1_1, y^1_2, y^2_1, y^2_2\right) \to \left(x^1, x^2, y^1, y^2, p^1_1, p^1_2, p^2_1, p^2_2\right), \tag{8.5}$$

where

$$p^1_1 = y^2_2, \qquad p^1_2 = y^1_2, \qquad p^2_1 = y^2_1, \qquad p^2_2 = y^1_1. \tag{8.6}$$

For the Hamiltonian we obtain

$$H = p^1_1 p^2_2 + p^1_2 p^2_1, \tag{8.7}$$

and Hamilton–De Donder equations (3.5) in Legendre coordinates take the form

$$\begin{aligned} &\frac{\partial p^1_1}{\partial x^1} + \frac{\partial p^2_1}{\partial x^2} = 0, \qquad \frac{\partial y^1}{\partial x^1} = p^2_2, \qquad \frac{\partial y^2}{\partial x^1} = p^2_1 \\ &\frac{\partial p^1_2}{\partial x^1} + \frac{\partial p^2_2}{\partial x^2} = 0, \qquad \frac{\partial y^1}{\partial x^2} = p^1_2, \qquad \frac{\partial y^2}{\partial x^2} = p^1_1. \end{aligned} \tag{8.8}$$

Now, we consider a π-adapted constraint in $J^1(\mathbb{R}^2 \times \mathbb{R}^2)$, defined by two constraint functions

$$f^1_1 = y^2_1 - g^1_1 = y^2_1 - y^1_1, \qquad f^1_2 = y^2_2 - g^1_2 = y^2_2 - y^1_2, \tag{8.9}$$

i.e., $k = 1$. This constraint satisfies the rank condition

$$\operatorname{rank}\left(\frac{\partial f^a_j}{\partial y^\sigma_i}\right) = \operatorname{rank}\begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} = 1, \tag{8.10}$$

where (a, j, i) label rows and (σ) label columns. This means that $\kappa = k = 1$, and by theorem 4.13 the constraint is Lagrangian.

We obtain one constraint form annihilating the canonical distribution $\mathcal{C}$; by (4.8) it reads

$$\varphi^1 = -\mathrm{d}y^1 + \mathrm{d}y^2. \tag{8.11}$$

Equivalently, the canonical distribution $\mathcal{C}$ is spanned by the following independent vector fields:

$$\frac{\partial_c}{\partial x^1} = \frac{\partial}{\partial x^1}, \qquad \frac{\partial_c}{\partial x^2} = \frac{\partial}{\partial x^2}, \qquad \frac{\partial_c}{\partial y^1} = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}, \qquad \frac{\partial}{\partial y^1_1}, \qquad \frac{\partial}{\partial y^1_2}. \tag{8.12}$$

Let us compute the constrained system. By theorem 5.5 we get

$$\bar{\alpha} = \bar{A}_1 \omega^1 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \sum_{i,j=1,2} \bar{B}^{ij}_{11} \omega^1 \wedge \mathrm{d}y^1_j \wedge \omega_i, \tag{8.13}$$

where

$$\bar{A}_1 = 0, \qquad \bar{B}^{ij}_{11} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}. \tag{8.14}$$

Hence, the constrained Euler–Lagrange equation is one second-order PDE

$$\frac{\partial^2 y^1}{\partial x^1 \partial x^2} = 0. \tag{8.15}$$

We can see that in this simple case the constrained Euler–Lagrange equation coincides with the (usual) Euler–Lagrange equation of the constrained Lagrange function

$$\bar{L} = L \circ \iota = 2 y^1_1 y^1_2. \tag{8.16}$$

Since $\det(\bar{B}^{ij}_{11}) \neq 0$, the constrained system is regular according to corollary 6.5, and the integrability condition (7.1) is satisfied. This means that we can find constraint Legendre transformation and express constrained Hamilton–De Donder equations in the canonical form. We obtain constraint momenta

$$P^1_1 = 2y^1_2, \qquad P^2_1 = 2y^1_1, \tag{8.17}$$

and energy 1-forms

$$\eta = \eta_1\,\mathrm{d}x^1 + \eta_2\,\mathrm{d}x^2 - \tfrac{1}{2}P^2_1\,\mathrm{d}P^1_1 - \tfrac{1}{2}P^1_1\,\mathrm{d}P^2_1 \quad \mathrm{mod}\ \mathcal{I}. \tag{8.18}$$

The class of energy 1-forms obviously contains a closed form, $\eta = -\mathrm{d}\bar{H}$, with

$$\bar{H} = \tfrac{1}{2}P^1_1 P^2_1. \tag{8.19}$$

Constrained Hamilton–De Donder equations consist of five first-order PDEs, including three field equations (for a field on the constraint submanifold)

$$\frac{\partial P^1_1}{\partial x^1} + \frac{\partial P^2_1}{\partial x^2} = 0, \qquad \frac{\partial y^1}{\partial x^1} = \frac{1}{2}P^2_1, \qquad \frac{\partial y^1}{\partial x^2} = \frac{1}{2}P^1_1, \tag{8.20}$$

and two equations of the constraint:

$$\frac{\partial y^2}{\partial x^1} = \frac{\partial y^1}{\partial x^1}, \qquad \frac{\partial y^2}{\partial x^2} = \frac{\partial y^1}{\partial x^2}. \tag{8.21}$$

Example 8.2. We shall give an example of a singular Lagrangian regularized by a π-adapted constraint.

Consider the fibred manifold $\pi : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ with canonical coordinates (x^i, y^σ), $1 \leqslant i, \sigma \leqslant 4$, and a first-order Lagrangian

$$L = \frac{1}{2}\left(\sum_{\sigma,j}\left(y^\sigma_j\right)^2\right) - \left(y^3_1\right)^2 - \left(y^4_4\right)^2. \tag{8.22}$$

In this case $A_\sigma = 0$, and the matrix B is singular (B is a diagonal matrix with two zero rows). Euler–Lagrange equations take the form

$$\begin{aligned}
&\frac{\partial^2 y^1}{(\partial x^1)^2} + \frac{\partial^2 y^1}{(\partial x^2)^2} + \frac{\partial^2 y^1}{(\partial x^3)^2} + \frac{\partial^2 y^1}{(\partial x^4)^2} = 0,\\
&\frac{\partial^2 y^2}{(\partial x^1)^2} + \frac{\partial^2 y^2}{(\partial x^2)^2} + \frac{\partial^2 y^2}{(\partial x^3)^2} + \frac{\partial^2 y^2}{(\partial x^4)^2} = 0,\\
&\frac{\partial^2 y^3}{(\partial x^2)^2} + \frac{\partial^2 y^3}{(\partial x^3)^2} + \frac{\partial^2 y^3}{(\partial x^4)^2} = 0,\\
&\frac{\partial^2 y^4}{(\partial x^1)^2} + \frac{\partial^2 y^4}{(\partial x^2)^2} + \frac{\partial^2 y^4}{(\partial x^3)^2} = 0.
\end{aligned} \tag{8.23}$$

Consider a constraint in $J^1(\mathbb{R}^4 \times \mathbb{R}^4)$ given by the following constraint functions:

$$\begin{aligned}
f^1_1 &= y^4_1 - g^1_1 = y^4_1 - \left(y^3_1\right)^2 - y^2_2 - y^1_2 - y^3_3,\\
f^1_2 &= y^4_2 - g^1_2 = y^4_2 - y^3_2 y^3_3 - y^3_3 y^3_4 - y^3_4 y^3_2,\\
f^1_3 &= y^4_3 - g^1_3 = y^4_3,\\
f^1_4 &= y^4_4 - g^1_4 = y^4_4.
\end{aligned} \tag{8.24}$$

This is a π-adapted constraint of rank $\kappa = 3$ (the rank of the matrix in (4.2) is equal to 3), and $k = 1$. By theorem 4.13 this constraint is not Lagrangian. Substituting into (4.8) we get the canonical distribution $\mathcal{C}$ annihilated by the following system of linearly independent constraint 1-forms φ^a, $a = 1$ and φ^α, $\alpha = 1, 2$:

$$\left(\left(y_1^3\right)^2 + y_2^2 + y_2^1 + y_3^3\right) \mathrm{d}x^1 + \left(y_2^3 y_3^3 + y_3^3 y_4^3 + y_4^3 y_2^3\right) \mathrm{d}x^2 - \mathrm{d}y^4, \quad \omega^2 + \omega^1, \quad \omega^3. \tag{8.25}$$

The explicit expression of functions $\bar{L}_a^j$, $1 \leqslant a \leqslant k$, $1 \leqslant j \leqslant n$, defined in (5.5), is

$$\bar{L}_1^1 = \left(y_1^3\right)^2 + y_2^2 + y_2^1 + y_3^3, \qquad \bar{L}_1^2 = y_2^3 y_3^3 + y_3^3 y_4^3 + y_4^3 y_2^3, \qquad \bar{L}_1^3 = 0, \qquad \bar{L}_1^4 = 0, \tag{8.26}$$

and of $\bar{L}_\alpha^j$, $1 \leqslant \alpha \leqslant \kappa - k$, $1 \leqslant j \leqslant n$, defined in (5.5) is

$$\bar{L}_1^1 = y_1^2, \quad \bar{L}_2^1 = 0, \quad \bar{L}_1^2 = y_2^2, \quad \bar{L}_2^2 = y_2^3, \quad \bar{L}_1^3 = y_3^2, \quad \bar{L}_2^3 = y_3^3, \quad \bar{L}_1^4 = y_4^2, \quad \bar{L}_2^4 = y_4^3. \tag{8.27}$$

Using relations (5.9) we obtain for $\mathcal{C}_{ri}^{aj}$, $1 \leqslant a \leqslant k$, $1 \leqslant r \leqslant m - \kappa$, that the only non-zero function is the following one,

$$\mathcal{C}_{11}^{12} = -1, \tag{8.28}$$

and for $\mathcal{C}_{ri}^{\alpha j}$, $1 \leqslant \alpha \leqslant \kappa - k$, $1 \leqslant r \leqslant m - \kappa$, the only non-zero functions are

$$\mathcal{C}_{11}^{11} = -1, \qquad \mathcal{C}_{12}^{12} = -1, \qquad \mathcal{C}_{13}^{13} = -1, \qquad \mathcal{C}_{14}^{14} = -1. \tag{8.29}$$

The matrix $\left(\tilde{B}_{rs}^{ij}\right)$ in (5.16) representing the constrained system takes the form

$$\tilde{B} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8.30}$$

and one can see that the problem is now regular since the regularity condition (6.5) is satisfied, i.e.,

$$\operatorname{rank}\left(\tilde{B}_{rs}^{ij}\right) = 4 = \max. \tag{8.31}$$

With the help of (5.24) we get one constrained Euler–Lagrange equation

$$y_{11}^1 - y_{11}^2 + y_{22}^1 - y_{22}^2 + y_{33}^1 - y_{33}^2 + y_{44}^1 - y_{44}^2 = 0. \tag{8.32}$$

Due to the regularity of the constrained system we have on Q the constraint Legendre transformation

$$\left(x^i, y^\sigma, y_1^1, y_2^1, y_3^1, y_4^1, y_i^2, y_i^3\right) \to \left(x^i, y^\sigma, P_1^1, P_1^2, P_1^3, P_1^4, y_i^2, y_i^3\right), \tag{8.33}$$

where constraint momenta P_r^i, $1 \leqslant r \leqslant m - \kappa$ (7.10) take the form

$$P_1^1 = y_1^1 - y_1^2, \qquad P_1^2 = y_2^1 - y_2^2, \qquad P_1^3 = y_3^1 - y_3^2, \qquad P_1^4 = y_4^1 - y_4^2. \tag{8.34}$$

For the inverse transformation we have

$$\left(x^i, y^\sigma, P_1^1, P_1^2, P_1^3, P_1^4, y_i^2, y_i^3\right) \to \left(x^i, y^\sigma, y_1^1, y_2^1, y_3^1, y_4^1, y_i^2, y_i^3\right), \tag{8.35}$$

where

$$y_1^1 = P_1^1 + y_1^2, \qquad y_2^1 = P_1^2 + y_2^2, \qquad y_3^1 = P_1^3 + y_3^2, \qquad y_4^1 = P_1^4 + y_4^2. \tag{8.36}$$

Now, using (7.6) we can compute the family of energy 1-forms expressed in constraint Legendre coordinates,

$$\eta = \eta_j \wedge \mathrm{d}x^j - \left(P_1^1 + y_1^2\right) \mathrm{d}P_1^1 - \left(P_1^2 + y_2^2\right) \mathrm{d}P_1^2 - \left(P_1^3 + y_3^2\right) \mathrm{d}P_1^3 - \left(P_1^4 + y_4^2\right) \mathrm{d}P_1^4 \quad \mathrm{mod}\ \mathcal{I}, \tag{8.37}$$

and the constrained Hamilton equations in the canonical form become

$$\begin{aligned} &\frac{\partial P_1^1}{\partial x^1} + \frac{\partial P_1^2}{\partial x^2} + \frac{\partial P_1^3}{\partial x^3} + \frac{\partial P_1^4}{\partial x^4} = 0, \\ &\frac{\partial y^1}{\partial x^1} = P_1^1 + y_1^2, \qquad \frac{\partial y^1}{\partial x^2} = P_1^2 + y_2^2, \\ &\frac{\partial y^1}{\partial x^3} = P_1^3 + y_3^2, \qquad \frac{\partial y^1}{\partial x^4} = P_1^4 + y_4^2. \end{aligned} \tag{8.38}$$

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